22.

ADNOTATIONES
AD
CALCULUM INTEGRALEM
EULERI

In quibus nonnulla Problemata ab EULERO *proposita resolvuntur*

AUCTORE

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TICINI.

EX TYPOGRAPHIA PETRI GALEATII
PRAESID. REI LITTER. PERMITT.
ANNO MDCCXC.

IOSEPHO . S. R. I. COMITI

DE . WILZECK

LEOPOLDI . II. REGIS

A . CVBICVLO . A . CONSILIIS . SANCTIORIBVS

PER . INSVBRIAM

PLENA . CVM . POTESTATE . ADMINISTRO

VIRO . OPTIMO . INTEGERRIMO

LIBELLVM . SVVM

COMMENTARIVM . IN . EVLERVM

GRATI . ANIMI . MONVMENTVM

LAVRENTIVS . MASCHERONIVS

D. D. L. M.

ADNOTATIONES
A D
CALCULUM INTEGRALEM
EULERI.

Adnotatio I. ad Cap. IV. Sect. I. Vol. I.

Determinatio constantis finitae in aequatione

$$\int \frac{dz}{lz} = \text{Const.} + llz + lz, \&c.$$

posito quod integrale annihiletur quando $z = 0$.

AD §. 219. docet Auctor integrationes pro variis casibus formulae differentialis $\frac{x^{m-1}\,dx}{(lx)^n}$, in quibus n est numerus integer positivus pendere „ a formula $\int \frac{x^{m-1}\,dx}{lx}$, „ quae posito $x^m = z$, ob $x^{m-1}\,dx = \frac{1}{m}\,dz$, & $lx = \frac{1}{m}\,lz$ „ reducitur ad hanc simplicissimam formam $\int \frac{dz}{lz}$, cuius in- „ tegrale si assignari posset, amplissimum usum in Analysi „ esset allaturum, verum nullis adhuc artificiis neque per logarithmos

„ garithmos neque per angulos exhiberi potuit ". Quomodo autem per feriem exprimi possit infra ostendit (§. 228.). Subdit vero. „ Videtur ergo haec formula $\int \frac{dz}{lz}$ singularem speciem functionum transcendentium suppeditare, quae utique „ accuratiorem evolutionem meretur. Eadem autem quantitas „ transcendens in integrationibus formularum exponentialium „ frequenter occurrit, quas in hoc capite tractare instituimus, „ propterea quod cum logarithmis tam arcte cohaeret, ut „ alterum genus facile in alterum converti possit: veluti ipsa „ formula modo considerata $\frac{dz}{lz}$ posito $lz = x$, ut fit $z = e^x$ „ & $dz = e^x dx$ transformatur in hanc exponentialem $\frac{e^x dx}{x}$, „ cuius ergo integratio aeque est abscondita ".

Citato vero §. 228. docet esse

$$\int \frac{e^x dx}{x} = C + lx + \frac{x}{1} + \frac{1}{2} \cdot \frac{x^2}{1.2} + \frac{1}{3} \cdot \frac{x^3}{1.2.3} + \frac{1}{4} \cdot \frac{x^4}{1.2.3.4} + \&c... \quad (1)$$

atque hinc

$$\int \frac{dz}{lz} = C + llz + \frac{lz}{1} + \frac{1}{2} \frac{(lz)^2}{1.2} + \frac{1}{3} \cdot \frac{(lz)^3}{1.2.3} + \frac{1}{4} \cdot \frac{(lz)^4}{1.2.3.4} + \&c... \quad (2)$$

deinde subdit „ quod integrale si debeat evanescere sumpto „ $z = 0$ constans C fit infinita unde pro reliquis casibus nihil concludi potest. Idem incommodum locum habet, si „ evanescens reddamus casu $z = 1$ quia $llz = l0$ fit infinitum. Caeterum patet si integrale sit reale pro valoribus „ ipsius z unitate minoribus, ubi lz est negativus; tum pro „ valoribus unitate maioribus fieri imaginarium, & vicissim. „ Hinc ergo (rursus concludit) natura huius functionis transcendentis parum cognoscitur ".

Quoniam ita fortasse impossibile est exhibere integrale $\int \frac{dz}{lz}$ per logarithmos aut per angulos, uti impossibile est exhibere

hibere integrale $\int \frac{dz}{z}$ per functionem algebraicam; inde dicendum erit hanc formulam $\int \frac{dz}{lz}$ singularem speciem functionum transcendentium suppeditare, quae accuratiorem evolutionem mereatur. Sed natura functionis transcendentis $\int \frac{dz}{z}$ satis cognosci censetur, quia licet non possit exprimi per functionem finitam algebraicam, tamen exprimitur per seriem, cuius summa est logarithmus z; quae series talis est, ut eius constans pro casu $z = 1$ possit determinari, & quae si ipsa convergens non sit pro aliquibus valoribus ipsius z, tamen possit in alias convergentes transformari, & quae demum exhibeat valores reales, quotiescumque tales valores competere debent formulae integrali $\int \frac{dz}{z}$. Eodem ergo modo etiam functio transcendens, quae oritur ex hac formula $\int \frac{dz}{lz}$ satis cognosci censenda erit, si per series saltem infinitas exhibeatur, quarum summa novo nomine, si cui libuerit erit appellanda, prout novum est etiam genus transcendentiae; quae series tales sint, ut constantes per integrationem ingressae possint determinari pro casu $z = 1$, aut pro casu $z = 0$, & quarum saltem aliqua semper convergens sit pro quocumque valore lz, & quae demum exhibeant valores reales, quotiescumque tales valores competere debent formulae integrali $\int \frac{dz}{lz}$. Nihil enim aliud requiri potest, quando ipsa formula per terminos finitos integrari nequeat ob novum transcendentiae genus. Simul ac vero haec omnia, quae commemorata sunt, praestita fuerint, tunc censebimus assignatum esse integrale huius formulae $\int \frac{dz}{lz}$ quod col-

locari poterit si libeat sub novo symbolo transcendentiae, sub quo si adhibeatur non minus ac functiones circulares, & logarithmi, amplissimum usum in Analysi erit allaturum.

Iam vero quemadmodum integrale formulae $\int \frac{dz}{z}$ est logarithmus ipsius z; ita integrale formulae $\int \frac{dz}{lz}$, quod Eulero videtur transcendens novi generis, appelletur si libet hyperlogarithmus ipsius z. Nos huiusmodi hyperlogarithmum, seu si novum nomen offenderit, nos huiusmodi integrale formulae $\int \frac{dz}{lz}$ ita assignabimus per varias series, ut primo constans pro his seriebus assignari possit pro casu $z = 0$. Secundo ut aliqua ex his seriebus satis convergens sit pro quocumque valore lz. Tertio ut hae series exhibeant valores reales pro quocumque valore reali lz, ac proinde pro quocumque valore reali formulae $\int \frac{dz}{lz}$. Quibus rebus tota huius functionis doctrina absolvetur.

Ac primo determinabimus valorem realem finitum C pro duabus aequationibus Euleri (1) & (2), si integrale debeat evanescere sumpto $z = 0$ (*).

Sit $C = E + \log. - F$; sit vero $A = E + \log. F$, erit $\int \frac{dz}{lz}$

(*) Valorem huius constantis finitum esse debere iam demonstraverat egregius iuvenis Thomas Rossi, qui Mathesim, & Philosophiam publice repetit in R. Ticinensi Archigymnasio, & in R. Ghisleriorum Coll. Suam demonstrationem simul cum aliis elegantibus animadversionibus circa hoc integrale mihi, dum haec praelo mandabantur, humaniter communicavit. Cum tamen ipse constantem minime determinaverit; id a nobis efficietur.

$$\int \frac{dz}{lz} = E + l - F + llz + lz + \frac{1}{2}\frac{(lz)^2}{2} + \frac{1}{3}\frac{(lz)^3}{2.3} + \&c.$$

$$= E + lF \quad + l - lz + lz + \frac{1}{2}\frac{(lz)^2}{2} + \frac{1}{3}\frac{(lz)^3}{2.3} + \&c.$$

$$= A + l - lz + lz \quad + \frac{1}{2}\frac{(lz)^2}{2} + \frac{1}{3}\frac{(lz)^3}{2.3} + \&c.$$

Modo cum fit $\int \frac{dz}{lz} = \frac{z}{lz} + \int \frac{dz}{(lz)^2}; \int \frac{dz}{(lz)^2} = \frac{z}{(lz)^2}$

$$+ 2\int \frac{dz}{(lz)^3};\ 2\int \frac{dz}{(lz)^3} = 2\,\frac{z}{(lz)^3} + 2.3\int \frac{dz}{(lz)^4}\ \&c.$$

habebimus

$$\int \frac{dz}{lz} = \frac{z}{lz} + \frac{z}{(lz)^2} + 2\,\frac{z}{(lz)^3} + 2.3\,\frac{z}{(lz)^4} + 2.3.4\,\frac{z}{(lz)^5} + \&c.$$

$$= A + l - lz + lz + \frac{(lz)^2}{2.\,2} + \frac{(lz)^3}{2.3.3} + \frac{(lz)^4}{2.3.4.4} + \&c.$$

Facile autem apparet feriei (3) nullam conftantem addendam effe pofito quod effe debeat $\int \frac{dz}{lz} = 0$ quando $z = 0$.

Iam vero cum pro quocumque cafu $z < 1$ quantitas $l - lz$ habeat fuum valorem realem; ipfa habebit hunc valorem a cafu $z = 0$ ufque ad cafum $z = e^{-1}$, fumpta pro e bafi logarithmica hyperbolica. Sed in cafu $z = e^{-1}$, $lz = -1$; $l - lz = 0$, ac proinde

$$A - 1 + \frac{1}{2.2} - \frac{1}{2.3.3} + \frac{1}{2.3.4.4} - \frac{1}{2.3.4.5.5} + \&c. \ldots$$

$$= e^{-1}(-1 + 1 - 1.2 + 1.2.3 - 1.2.3.4 + 1.2.3.4.5 - \&c. \ldots).$$

Sit fumma feriei $-1 + \frac{1}{2.2} - \frac{1}{2.3.3} + \frac{1}{2.3.4.4} - \&c. = -L$ fumma feriei $1 - 1.2 + 1.2.3 - 1.2.3.4 + \&c. = M$; habebitur $A + e^{-1} - L = e^{-1}M$; $A = L + e^{-1}(M - 1)$. Eulerus autem in Calculo Diff. Part. Poft. Cap. I. §. 10. afferit

fe in-

se inveniſſe $M = 0,4036524077$ ſine errore in ipſa ultima cyphra; facile vero invenitur eſſe $L = 0,796599 599297 053134 283\ldots$ Quare cum ſit $e = 2,718281 828459 045235\ldots$; erit $A = 0,577215 5802\ldots$ Revera tamen oſtendemus eſſe $A = 0,577215 6649\ldots$, cum error obrepſerit ob errorem in ultimis aliquot cyphris numeri $0,4036524077$.

Iam manifeſte liquet quod cum duae ſeries (3) & (4) habeant idem differentiale $\frac{dz}{lz}$; & cum per valorem inventum $A = 0,577215\ldots$ aequentur inter ſe pro caſu $z = e^{-1}$, aequales erunt etiam pro quocumque alio valore z a caſu $z = e^{-1}$ uſque ad caſum $z = 0$. Quare cum in caſu $z = 0$ annihiletur ſeries (3), annihilabitur etiam ipſi aequalis (4). Cum vero ſit $A = E + lF$; $C = E + l - F$; erit conſtans Euleri $C = A - lF + l - F = A + l - 1$.

Non erit abs re invenire eundem valorem $A = 0,577215\ldots$ etiam alia via, unde habebitur modus corrigendi errores illos in cyphris ultimis ſupra notatos; idque eo magis praeſtare iuvabit cum numerus $0,577215 664901 532\ldots$ qui prodit pro valore A ſit aliunde cognitus in Analyſi, qui vix haberetur ad plures cyphras ope ſummationis ſeriei $1 - 1.2 + 1.2.3 - 1.2.3.4 + \&c.\ldots$ per methodos communes.

Poſito $z = e^x$; $lz = x$; $dz = e^x dx$ habetur

$$\int \frac{dz}{lz} = \int \frac{e^x dx}{x} = \int \frac{dx}{x}\left(1 + x + \frac{x^2}{2} + \frac{x^3}{2.3} + \frac{x^4}{2.3.4} + \&c.\right.$$

$$= A + l - x + x + \frac{x^2}{2.2} + \frac{x^3}{2.3.3} + \frac{x^4}{2.3.4.4} + \&c.\ldots \quad (5)$$

ubi negotium non faceſſat terminus $l - x$ loco lx; tum quia uterque habetur eodem iure per integrationem formulae differentialis $\frac{dx}{x} = \frac{-dx}{-x}$; tum quia ipſe terminus lx abit in $l - x$ per mutationem conſtantis A uti ſupra vidimus. Habetur item.

$e^x dx$

$$\int\frac{e^x dx}{x}=\frac{e^x}{x}+\int\frac{e^x dx}{x^2}=\frac{e^x}{x}+\int\frac{dx}{x^2}\left(1+x+\frac{x^2}{2}+\frac{x^3}{2.3}+\&c\ldots\right)$$

$$=\frac{e^x}{x}+B-\frac{1}{x}+l-x+\frac{x}{2}+\frac{x^2}{2.2.3}+\frac{x^3}{2.3.3.4}+\ \&c\ldots\ (6)$$

ubi B eſt alia conſtans ingreſſa per integrationem. Definitur autem ea conſtans reſpectu A hoc modo. Evolvatur terminus $\frac{e^x}{x}$, qui reperitur in hac ultima ſerie per valorem $e^x = 1+x+\frac{x^2}{2}+$ &c. Habebitur in ſerie inde enaſcente terminus conſtans 1. Caeteri omnes termini numero infiniti afficientur variabili x. Erit ergo addendo hunc terminum conſtantem ipſi B; $\int\frac{e^x dx}{x}=B+1+Q+l-x$; ubi Q includit omnes terminos algebraicos continentes x tam natos ex evolutione $\frac{e^x}{x}$, quam poſitos ante & poſt $l-x$ in ſerie (6). Includendo autem in P terminos omnes algebraicos poſitos in ſerie (3) poſt $l-x$, debet eſſe $B+1+Q+l-x = A+l-x+P$. Cum ergo Q debeat idem eſſe cum P, cum ſint ſeries earundem poteſtatum ipſius x; erit $B+1=A$; $B=A-1$.

Eſt item $\int\frac{e^x dx}{x}=\frac{e^x}{x}+\frac{e^x}{x^2}+2\int\frac{e^x dx}{x^3}=\frac{e^x}{x}+\frac{e^x}{x^2}$

$$+2\int\frac{dx}{x^3}\left(1+x+\frac{x^2}{2}+\frac{x^3}{2.3}+\&c\ldots\right)=\frac{e^x}{x}+\frac{e^x}{x^2}+C-$$

$\frac{1}{x^2}-\frac{2}{x}+l-x+\frac{x}{3}+\frac{x^2}{2.3.4}+$ &c.... (7) ubi item C eſt alia conſtans ingreſſa per integrationem. Determinatur autem C reſpectu B hoc modo. Cum debeat eſſe (6) = (7); ac proinde

proinde $B - \frac{1}{x} + l - x + \frac{x}{2} + \frac{x^2}{2.2.3} + \&c. \ldots = \frac{e^x}{x^2} + C$
$- \frac{1}{x^2} - \frac{2}{x} + l - x + \frac{x}{3} + \frac{x^2}{2.3.4} + \&c. \ldots$; si ipsi C addatur quantitas constans $\frac{1}{2}$ orta ex evolutione $\frac{e^x}{x^2}$ debebit esse $C + \frac{1}{2} = B$; cum caeteri omnes termini affecti x debeant esse iidem in utraque serie; erit ergo $C = B - \frac{1}{2} = A - 1 - \frac{1}{2}$.

Est item in genere $\int \frac{e^x dx}{x} = \frac{e^x}{x} + \frac{e^x}{x^2} + 2\frac{e^x}{x^3} + 2.3\frac{e^x}{x^4} + \ldots$
$+ 2.3.4\ldots(n-1)\frac{e^x}{x^n} + 2.3.4\ldots n \int \frac{e^x dx}{x^{n+1}}$, ubi n indicat numerum terminorum, qui habentur ante terminum summatorium $2.3.4\ldots\ldots n \int \frac{e^x dx}{x^{n+1}}$. Ac proinde erit

$$\int \frac{e^x dx}{x} = \frac{e^x}{x} + \frac{e^x}{x^2} + 2\frac{e^x}{x^3} + 2.3\frac{e^x}{x^4} + \ldots + 2.3.4\ldots(n-1)\frac{e^x}{x^n}$$
$$+ 2.3.4\ldots\ldots n \int \frac{dx}{x^{n+1}} \left(1 + x + \frac{x^2}{2} + \frac{x^3}{2.3} + \frac{x^4}{2.3.4} + \ldots\ldots\right)$$
$$= \frac{e^x}{x} + \frac{e^x}{x^2} + 2\frac{e^x}{x^3} + 2.3\frac{e^x}{x^4} + \ldots\ldots + 2.3.4\ldots(n-1)\frac{e^x}{x^n}$$
$$+ F - \frac{2.3.4\ldots.n}{nx^n} - \frac{2.3.4\ldots.n}{(n-1)x^{n-1}} - \frac{2.3.4\ldots.n}{2(n-2)x^{n-2}} -$$
$$\frac{2.3.4\ldots.n}{2.3(n-3)x^{n-3}} - \ldots\ldots - \frac{n}{x} + l - x + \frac{x}{n+1} + \frac{x^2}{2(n+1)(n+2)}$$
$$+ \frac{x^3}{3(n+1)(n+2)(n+3)} + \&c.\ldots (8),$$
ubi F est constans ingressa per integrationem.

Et po-

Et ponendo $n+1$ loco n erit denuo

$$\int\frac{e^x dx}{x}=\frac{e^x}{x}+\frac{e^x}{x^2}+2\frac{e^x}{x^3}+2.3\frac{e^x}{x^4}+\ldots\ldots 2.3.4\ldots(n-1)\frac{e^x}{x^n}$$

$$+2.3.4\ldots\ldots n\frac{e^x}{x^{n+1}}+2.3.4\ldots(n+1)\int\frac{dx}{x^{n+2}}\left(1+x+\frac{x^2}{2}\ldots\right)$$

$$=\frac{e^x}{x}+\frac{e^x}{x^2}+2\frac{e^x}{x^3}+2.3\frac{e^x}{x^4}+\ldots\ldots+2.3.4\ldots(n-1)\frac{e^x}{x^n}$$

$$+2.3.4\ldots n\frac{e^x}{x^{n+1}}+G-\frac{2.3.4\ldots(n+1)}{(n+1)x^{n+1}}-\frac{2.3.4\ldots(n+1)}{nx^n}$$

$$-\frac{2.3.4\ldots(n+1)}{2(n-1)x^{n-1}}-\frac{2.3.4\ldots(n+1)}{2.3.(n-2)x^{n-2}}-\ldots\ldots-\frac{n+1}{x}+l-x$$

$$+\frac{x}{n+2}+\frac{x^2}{2(n+2)(n+3)}+\frac{x^3}{3(n+2)(n+3)(n+4)}+\&c.\ldots (9)$$

ubi etiam est G quantitas constans adiecta per integrationem. Determinatur autem G respectu F ex aequatione, quae haberi debet $(8)=(9)$, unde oritur $F-\frac{2.3.4\ldots n}{nx^n}-$

$$\frac{2.3.4\ldots n}{(n-1)x^{n-1}}-\frac{2.3.4\ldots n}{2(n-2)x^{n-2}}-\frac{2.3.4\ldots n}{2.3(n-3)x^{n-3}}-\ldots\ldots-\frac{n}{x}$$

$$+l-x+\frac{x}{n-1}+\frac{x^2}{2(n+1)(n+2)}+\frac{x^3}{3(n+1)(n+2)(n+3)}+\&c.\ldots=$$

$$2.3.4\ldots n\frac{e^x}{x^{n+1}}+G-\frac{2.3.4\ldots(n+1)}{(n+1)x^{n+1}}-\frac{2.3.4\ldots(n+1)}{nx^n}-$$

$$\frac{2.3.4\ldots(n+1)}{2(n-1)x^{n-1}}-\frac{2.3.4\ldots(n+1)}{2.3(n-2)x^{n-2}}-\ldots\ldots-\frac{n+1}{x}+l-x$$

$$+\frac{x}{n+2}+\frac{x^2}{2(n+2)(n+3)}+\frac{x^3}{3(n+2)(n+3)(n+4)}+\&c.\ldots$$

Evoluto itaque termino $2.3.4\ldots n\frac{e^x}{x^{n+1}}$, & addito ipsi

G termino conſtanti $\frac{1}{n+1}$, qui inde oritur, habebimus.

$G+\frac{1}{n+1}=F$; $G=F-\frac{1}{n+1}$. Erit itaque ipſa conſtans

$F=A-1-\frac{1}{2}-\frac{1}{3}-\frac{1}{4}-\ldots\ldots-\frac{1}{n}$; atque aequatio (8)

ita exprimetur (10)....$\int\frac{e^x dx}{x}=e^x\left[\frac{1}{x}+\frac{1}{x^2}+2\frac{1}{x^3}+2.3\frac{1}{x^4}\right.$

$\left.+\ldots.+2.3.4\ldots(n-1)\frac{1}{x^n}\right]+A-1-\frac{1}{2}-\frac{1}{3}-\frac{1}{4}-\ldots.-\frac{1}{n}$

$-\frac{2.3.4\ldots.n}{nx^n}-\frac{2.3.4\ldots.n}{(n-1)x^{n-1}}-\frac{2.3.4\ldots.n}{2(n-2)x^{n-2}}-\frac{2.3.4\ldots.n}{2.3(n-3)x^{n-3}}$

$-\frac{(n-2)(n-1)n}{3x^3}-\frac{(n-1)n}{2x^2}-\frac{n}{x}+l-x+\frac{x}{n+1}+\frac{x^2}{2(n+1)(n+2)}$

$+\frac{x^3}{3(n+1)(n+2)(n+3)}+\&c.\ldots..$

Si nunc ſumatur $x=-n=-\infty$; annihilabitur ſeries poſita ante conſtantem A, atque multiplicata per e^x; atque ſumendo ſeries ex ordine ad dexteram atque ad ſiniſtram $l-x$ habebimus $\int\frac{e^x dx}{x}=A-1-\frac{1}{2}-\frac{1}{3}-\frac{1}{4}-\ldots.-\frac{1}{n}+l-x$

$$+\left\{\begin{array}{l}\frac{x}{n+1}+\frac{x^2}{2(n+1)(n+2)}+\frac{x^3}{3(n+1)(n+2)(n+3)}+\&c.\ldots\\ -\frac{n}{x}-\frac{(n-1)n}{2x^2}-\frac{(n-2)(n-1)n}{3x^3}-\&c.\ldots\end{array}\right.$$

Cum vero ob $x=-n=-\infty$ ſit $\frac{x}{n+1}=\frac{n}{x}$; $\frac{x^2}{2(n+1)(n+2)}=\frac{(n-1)n}{2x^2}$; $\frac{x^3}{3(n+1)(n+2)(n+3)}=\frac{(n-2)(n-1)n}{3x^3}$, &c.; remanebit

$\int\frac{e^x dx}{x}$

$$\int \frac{e^x dx}{x} = A - 1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \ldots - \frac{1}{n} + l - x.$$

Sed ex demonstratione Euleri in Calc. Differ. Part. Post. C. VI. est

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n} = ln + \frac{1}{2n} - \frac{\mathfrak{A}}{2n^2} + \frac{\mathfrak{B}}{4n^4} - \frac{\mathfrak{C}}{6n^6} + \frac{\mathfrak{D}}{8n^8} - \ldots$$
$$+ \frac{1}{2} + \frac{\mathfrak{A}}{2} - \frac{\mathfrak{B}}{4} + \frac{\mathfrak{C}}{6} - \frac{\mathfrak{D}}{8} + \ldots$$

ubi $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, &c. sunt numeri Bernoulliani. In casu autem $n = \infty$ est

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n} = ln + \frac{1}{2} + \frac{\mathfrak{A}}{2} - \frac{\mathfrak{B}}{4} + \frac{\mathfrak{C}}{6} - \frac{\mathfrak{D}}{8} + \ldots$$

Erit ergo

$$\int \frac{e^x dx}{x} = A - ln - \frac{1}{n} - \frac{\mathfrak{A}}{2} + \frac{\mathfrak{B}}{4} - \frac{\mathfrak{C}}{6} + \frac{\mathfrak{D}}{8} - \ldots + l - x$$

seu ob $ln = l - x$; $\int \frac{e^x dx}{x} = A - \frac{1}{2} - \frac{\mathfrak{A}}{2} + \frac{\mathfrak{B}}{4} - \frac{\mathfrak{C}}{6} + \frac{\mathfrak{D}}{8} -$ &c.

Quod integrale cum in casu ipso $x = -n = -\infty$ debeat annihilari; erit tandem $A = \frac{1}{2} + \frac{\mathfrak{A}}{2} - \frac{\mathfrak{B}}{4} + \frac{\mathfrak{C}}{6} - \frac{\mathfrak{D}}{8} +$ &c. $\ldots =$ 0, 577215 664901 5235, ut Eulerus invenit loco citato. Eadem Euleri methodo actu collectis centum terminis seriei $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}$ &c. invenimus esse

A = 0, 577215 664901 532860 618112 090082 39.

Cum vero sit $M = e(A - L) + 1$ erit $M = 1 - 1.2 + 1.2.3 -$ &c. $=$ 0, 403652 637676 805925 7....

Cum valor $l - lz$ sit realis a valore $z = 0$ usque ad valorem $z = 1 - \omega$, ubi ω est quantitas infinitesima; erit etiam intra hos limites

$$\int \frac{dz}{lz} = A + l - lz + lz + \frac{(lz)^2}{2.2} + \frac{(lz)^3}{2.3.3} + \frac{(lz)^4}{2.3.4.4} + \text{\&c.} \ldots$$

ubi nihil intercedet imaginarium.

 Hinc

Hinc evidens fit valorem $\int \frac{dz}{lz}$, qui in caſu $z=0$ annihilatur, pro caſu $z=1-\omega$ eſſe infinitum, quod demonſtravit nuper celeber. P. Gregorius Fontana, qui non ſatis perſpicuam invenit demonſtrationem Euleri. In caſu enim $z=1-\omega$ habemus $lz=-\omega$, & $\int \frac{dz}{lz}=A+l\omega=-\infty$.

Cum valor ipſius z eſt propior unitati, tunc commode adhibetur ſeries (4), quae eo magis convergit, quo minus z diſtat ab unitate. Cum vero valor ipſius z ſit propior zero, tunc adhibenda eſt ſeries (10) transformata, ut ſequitur.

Sumatur $n=-x+r$, ubi r ſit fractio poſitiva aut negativa talis ut ſit n proximior valori ipſius $-x$, erit $l-x=l(n-r)=ln-\frac{r}{n}-\frac{r^2}{2n^2}-\frac{r^3}{3n^3}-\&c.$, & aequatio (10) collatis in unum duabus ſeriebus hinc & inde ad latera termini $l-x$, abibit in ſequentem

$$\int \frac{e^x dx}{x}=e^x\left(\frac{1}{x}+\frac{1}{x^2}+2\frac{1}{x^3}+2.3\frac{1}{x^4}+2.3.4\frac{1}{x^5}+\dots+2.3.4..(n-1)\frac{1}{x^n}\right)$$
$$+A-1-\frac{1}{2}-\frac{1}{3}-\frac{1}{4}-\dots-\frac{1}{n}+ln-\frac{r}{n}-\frac{r^2}{2n^2}-\frac{r^3}{3n^3}-\dots$$
$$+\begin{cases}\frac{x}{n+1}+\frac{x^2}{2(n+1)(n+2)}+\frac{x^3}{3(n+1)(n+2)(n+3)}+\dots\\ -\frac{n}{x}-\frac{(n-1)n}{2x^2}-\frac{(n-2)(n-1)n}{3x^3}-\dots\end{cases}$$

ſeu

$$\int \frac{e^x dx}{x}=e^x\left(\frac{1}{x}+\frac{1}{x^2}+2\frac{1}{x^3}+2.3\frac{1}{x^4}+2.3.4\frac{1}{x^5}+\dots+2.3.4..(n-1)\frac{1}{x^n}\right)$$
$$-\frac{1}{2n}+\frac{A}{2n^2}-\frac{B}{4n^4}+\frac{C}{6n^6}-\frac{D}{8n^8}+\dots-\frac{r}{n}-\frac{r^2}{2n^2}-\frac{r^3}{3n^3}-\dots$$
$$+\left\{\frac{x}{n+1}\right.$$

$$+\begin{cases} \dfrac{x}{n+1}+\dfrac{x^2}{2(n+1)(n+2)}+\dfrac{x^3}{3(n+1)(n+2)(n+3)}+\ldots. \\ -\dfrac{n}{x}-\dfrac{(n-1)n}{2x^2}-\dfrac{(n-2)(n-3)}{3x^3}-\ldots. \end{cases}$$

ſeu tandem adhibitis aequationibus $x=lz$; $e^x=z$; habebitur

$$(\mathrm{V})=\int\frac{dz}{lz}=z\left(\frac{1}{lz}+\frac{1}{(lz)^2}+2\,\frac{1}{(lz)^3}+2.3\,\frac{1}{(lz)^4}+\ldots+2.3.4\ldots(n-1)\frac{1}{(lz)^n}\right)$$

$$-\frac{1}{2n}+\frac{\mathbf{A}}{2n^2}-\frac{\mathbf{B}}{4n^4}+\frac{\mathbf{C}}{6n^6}-\frac{\mathbf{D}}{8n^8}+\ldots-\frac{r}{n}-\frac{r^2}{2n^2}-\frac{r^3}{3n^3}-\ldots$$

$$+\begin{cases} \dfrac{lz}{n+1}+\dfrac{(lz)^2}{2(n+1)(n+2)}+\dfrac{(lz)^3}{3(n+1)(n+2)(n+3)}+\ldots. \\ -\dfrac{n}{lz}-\dfrac{(n-1)n}{2(lz)^2}-\dfrac{(n-2)(n-1)}{3(lz)^3}-\ldots. \end{cases}$$

in qua ſerie ex pluribus convergentibus compoſita ſinguli termini ſunt unitate minores excepto forte termino unico $\frac{n}{lz}$, qui tamen eſt minor 2.

Manifeſtum vero eſt ſeries adiectas poſt primam in aequatione (V) ortas ex evolutione termini $2.3.4\ldots.n\int\frac{e^x\,dx}{x^{n-1}}$ non eſſe contemnendas; atque adeo nunquam adhibendam eſſe aequationem (3) ſine additione, quando eius ſumma quaeritur proxime per realem ſummam plurium terminorum, quamvis ſine additione tunc adhiberi poſſit cum tota illa ſeries $\frac{z}{lz}+\frac{z}{(lz)^2}+$ &c. divergit, uti ſupra factum eſt in caſu $z=e^{-1}$; tunc enim revera non ſumantur termini, ſed per alias methodos peculiares quaeritur illa quantitas, ex qua enata eſt illa ſeries, quae improprie appellatur ſumma ſeriei divergentis. Eo igitur caſu adhiberi poteſt ſeries ſine additione. Tunc enim adhibendo duntaxat priores aliquot terminos

illius

illius seriei per methodos illas peculiares supra commemoratas iam non negliguntur sequentes termini; uti fit in summatione seriei convergentis per additionem aliquot terminorum; sed habetur ratio etiam reliquorum omnium, quae in cursu seriei ponenda sunt; investigatur enim quantitas ipsa, ex qua termini illi adhibiti cum omnibus consectariis enati sunt. Quod probe notandum erat.

Iam ergo pro casibus omnibus inter valorem $z = 0$, & $z = 1 - \omega$ ubi ω est quantitas infinitesima, praestitimus illa tria, quae nobis ab initio proposita fuerant, scilicet primo invenimus constantem addendam pro casu $z = 0$. Secundo assignavimus series quarum nonnulla semper convergat pro quocumque valore z intra hos limites. Tertio hae series semper reales inventae sunt pro his casibus, in quibus etiam invenitur semper realis valor quantitatis differentialis $\frac{dz}{lz}$.

Superest ut pergamus ad casus omnes qui continentur intra limites valorum $z = 1 + \omega$ usque ad $z = \infty$. Cum enim etiam pro his casibus quantitas differentialis $\frac{dz}{lz}$ sit realis; videndum est quodnam integrale sortiatur.

Celeberrimo Auctori visum est, ut supra retulimus, „ quod si integrale sit reale pro valoribus ipsius z unitate „ minoribus, tum pro valoribus unitate maioribus fiat ima- „ ginarium, & vicissim “. Nos sequentia notabimus.

Primo cum differentiale $\frac{dz}{lz}$ sit negativum pro valoribus z unitate minoribus, & cum ex negativo transeat in positivum cum z transit a valoribus unitate minoribus ad valores unitate maiores, quin tamen unquam hoc differentiale $\frac{dz}{lz}$ fiat imaginarium a valore $z = 0$ usque ad valorem $z = \infty$; dico quod

quod si quantitas constans, quae ingreditur per integrationem, sit realis pro valoribus z unitate minoribus, atque adeo totum integrale sit reale pro his valoribus; non poterit fieri ut evadat imaginarium pro valoribus z unitate maioribus. Etenim fluxio realis continua non posset quasi per saltum habere fluens imaginarium . . . .

Cum itaque in casu $z = 1 - \omega$ habeatur $\int \frac{dz}{lz} = A + l - l(1-\omega) = A + l + l(1+\omega)$, ob $l(l-\omega) = -\omega$, & $l(1+\omega) = +\omega$, & integrale idem esse debeat pro casu $z = 1 + \omega$, ac pro casu $z = 1 - \omega$ cum fluxio in hoc transitu infinitesimo non evaserit imaginaria; habebimus pro casu $z = 1 + \omega$; $\int \frac{dz}{lz} = A + l + lz$, atque adeo pro casibus omnibus a casu $z = 1 + \omega$, usque ad casum $z = \infty$ habebimus

$$\int \frac{dz}{lz} = A + l + lz + lz + \frac{(lz)^2}{2.2} + \frac{(lz)^3}{2.3.3} + \frac{(lz)^4}{2.3.4.4} + \&c. \ldots (11)$$

quae adhuc series tota est realis, & respondet valori semper reali quantitatis differentialis $\frac{dz}{lz}$.

Idem etiam demonstratur hoc modo. Cum sit

$$\int \frac{e^x dx}{x} = \frac{e^x}{x} + \frac{e^x}{x^2} + 2\frac{e^x}{x^3} + 2.3\frac{e^x}{x^4} + 2.3.4\frac{e^x}{x^5} + \&c. \text{ in infin.} \quad (12)$$

cui seriei nulla constans additur cum annihilari debeat pro casu $x = lz = -\infty$, seu pro casu $z = 0$, & cum haec series exhibeat valorem integralis $\int \frac{e^x dx}{x}$ ab ipso valore $x = lz = -\infty$ ad valorem $x = \infty$, seu a valore $z = 0$ ad valorem $z = \infty$; est etiam per aequationem (10).

$$\int \frac{e^x dx}{x} = e^x \left(\frac{1}{x} + \frac{1}{x^2} + 2\frac{1}{x^3} + 2.3\frac{1}{x^4} + \ldots + 2.3.4 \ldots (n-1)\frac{1}{x^n} \right) + A'$$

$$+A' - 1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \ldots - \frac{1}{n} - \frac{2.3.4\ldots n}{nx^n}$$

$$- \frac{2.3.4\ldots n}{(n-1)x^{n-1}} - \frac{2.3.4\ldots n}{2(n-2)x^{n-2}} - \frac{2.3.4\ldots n}{2.3(n-3)x^{n-3}}$$

$$- \ldots - \frac{(n-2)(n-1)n}{3x^3} - \frac{(n-1)n}{2x^2} - \frac{n}{x} + l - x$$

$$+ \frac{x}{n+1} + \frac{x^2}{2(n+1)(n+2)} + \frac{x^3}{3(n+1)(n+2)(n+3)} + \&c\ldots (13).$$

ubi supponimus non esse notum valorem constantis A' addendae quando x est quantitas positiva, atque addidimus apicem ut distingueremus ab A, quae constans inventa est supra pro casibus in quibus x est quantitas negativa. Sumamus nunc $x = n = \infty$. Ex collatione aequationum (12), & (13), & omissis in aequatione (13) terminis, qui se mutuo destruunt habebitur $A' - 1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \ldots - \frac{1}{n} + l - x = 0$;

$A' + l - 1 + lx - 1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \ldots - \frac{1}{n} = 0$, idest

$$A' + l - 1 + ln - ln - A = 0;\ A' = A - l - 1.$$

Quae constans ita inventa pro casibus, in quibus x habet valorem positivum, seu in quibus z est unitate maior si substituatur in aequatione (5) loco A; erit

$$\int \frac{e^x dx}{x} = A - l - 1 + l - x + x + \frac{x^2}{2.2} + \frac{x^3}{2.3.3} + \&c\ldots$$

$$= A + lx + x + \frac{x^2}{2.2} + \frac{x^3}{2.3.3} + \&c\ldots,\ \text{seu}$$

$$\int \frac{dz}{lz} = A + llz + lz + \frac{(lz)^2}{2.2} + \frac{(lz)^3}{2.3.3} + \&c\ldots$$

prorsus ut supra. Erit itaque in genere

$$\int \frac{dz}{lz} = A + l \mp lz + lz + \frac{(lz)^2}{2.2} + \frac{(lz)^3}{2.3.3} + \&c\ldots$$

ubi

ubi signum — adhibendum erit pro valoribus z minoribus unitate; signum vero $+$ pro valoribus eiusdem z unitate maioribus.

Videtur itaque hic in integratione logarithmica $\frac{dx}{x}$ adhibendum esse signum duplex $\mp$ ante x hoc modo $l \mp x$ fere uti in extractione radicum parium; quod novis exemplis in sequentibus uberius confirmabimus.

Pro valoribus z non admodum unitate maioribus praestabit uti serie $A + llz + lz + \frac{(lz)^2}{2.2} +$ &c.......; pro valoribus vero ingentibus ipsius z commodior erit series (V) iuxta methodos supra expositas.

Iam ergo habemus series, quarum aliqua semper est convergens, & quae tribuunt valores reales pro integrali $\int \frac{dz}{lz}$ etiam pro omnibus valoribus ipsius z unitate maioribus usque in infinitum, adeo ut cum $z = \infty$ fit $\int \frac{dz}{lz} = \frac{z}{lz}$. Quare iam omnia praestitimus, quae supra polliciti sumus pro omnibus valoribus ipsius z a zero usque ad infinitum, pro quibus est realis quantitas differentialis $\frac{dz}{lz}$; adeo ut huius integrale quamvis habitum per series, conferi debeat satis cognitum in Analysi; atque ad omnes usus, ad quos antea desiderabatur deinceps satis commode possit adhiberi.

 Usus

Usus integralis $\int \frac{dz}{lz}$ *supra determinati in ulterioribus integralibus.*

PRaeter usus, quos habet integrale superius determinatum in integrationibus ab Eulero commemoratis; alios etiam plurimos habere potest in novis quibusdam integrationibus, quarum aliquod specimen hic exhibebimus.

Proponatur exempli caussa integranda formula differentialis $dzllz$; habebimus $\int dzllz = zllz - \int \frac{dz}{lz} = zllz - A - llz - lz - \frac{(lz)^2}{2.2} - \&c..$; ubi cum $zllz$ annihiletur in casu $z=0$; in eodem casu totum integrale annihilabitur existente $A=0,\ 577215....$

In casu $z=1$ erit $\int dzllz = -A$, qui casus addi poterit capiti VIII. huius sectionis cui titulus: *De valoribus integralium, quos certis tantum casibus recipiunt.*

Secundo proponatur integranda formula $\frac{dz}{llz}$, cuius integrale appellari posset hypersecundus logarithmus z; sit $lz = x$; $z = e^x$; $llz = lx = y$; $x = e^y$; erit $z = e^{e^y}$; $dz = e^{e^y} e^y\, dy$; & $\int \frac{dz}{llz}$

$$= \int \frac{e^{e^y} e^y\, dy}{y} = \int \frac{e^y\, dy}{y} + \int \frac{e^{2y}\, dy}{y} + \tfrac{1}{2} \int \frac{e^{3y}\, dy}{y} + \frac{1}{2.3} \int \frac{e^{4y}\, dy}{y} + \frac{1}{2.3.4} \int \frac{e^{5y}\, dy}{y} + \&c... = \int \frac{dx}{lx} + \int \frac{2x dx}{lx^2} + \tfrac{1}{2} \int \frac{3x^2 dx}{lx^3} + \frac{1}{2.3} \int \frac{4x^3 dx}{lx^4} + \frac{1}{2.3.4} \int \frac{5x^4 dx}{lx^5} + \&c...$$

cuius seriei terminus generalis est $\frac{1}{2.3.4....(n-1)} \int \frac{nx^{n-1} dx}{lx^n}$;

qui

qui posito $x^n = u$ fit $\frac{1}{2.3.4.\ldots.(n-1)} \int \frac{du}{lu}$; cuius summa per seriem infinitam superius est tradita eo modo ut annihiletur posito $u = 0$, seu $x = 0$. Si ergo $\int \frac{du}{lu}$ liceat iam appellare hyperlogarithmum u, ac indicare hoc modo $\int \frac{du}{lu} = l'u$; sit vero hyperlogarithmus secundus $z = \int \frac{dz}{llz} = l''z$; erit

$$\int \frac{dz}{llz} = l''z = l'x + l'x^2 + \frac{1}{2} l'x^3 + \frac{1}{2.3} l'x^4 + \frac{1}{2.3.4} l'x^5 + \text{\&c.}$$

seu

$$\int \frac{dz}{llz} = l''z = l'lz + l'(lz)^2 + \frac{1}{2} l'(lz)^3 + \frac{1}{2.3} l'(lz)^4 + \frac{1}{2.3.4} l'(lz)^5 + \text{\&c.}\ldots,$$

quae series sine additione constantis fit $= 0$ quando $x = 0$; seu quando $z = 1$.

Observationes in triplicem modum integrandi formulam $dy = \frac{dx}{(lx)^n}$

Primo conferatur formula $dy = \frac{dx}{(lx)^n}$ cum formula Euleri $dy = \frac{X\,dx}{(lx)^n}$ §. 215. Cum hic sit $X = 1$; atque adeo $d.(Xx) = Pdx = dx$; erit $P = 1$; $Q = 1$; $R = 1$, unde sequitur

1°. $y = -x\left(\frac{1}{(n-1)(lx)^{n-1}} + \frac{1}{(n-1)(n-2)(lx)^{n-2}} + \frac{1}{(n-1)(n-2)(n-3)(lz)^{n-3}} + \text{\&c.}\right.$

adeo ut si sit n numerus integer positivus; integratio dedu-

catur ad formulam $\frac{1}{(n-1)(n-2)\ldots\ldots 1}\int\frac{dx}{lx}$, ut Eulerus docet loco citato.

Secundo cum sit $\int\frac{dx}{(lx)^n}=\frac{x}{(lx)^n}+n\int\frac{dx}{(lx)^{n+1}}$; erit

II°. $y=x\left(\frac{1}{(lx)^n}+\frac{n}{(lx)^{n+1}}+\frac{n(n+1)}{(lx)^{n+2}}+\frac{n(n+1)(n+2)}{(lx)^{n+3}}+\&c.\ldots\right)$,

quae series finita est quoties n erit numerus integer negativus, atque haec integratio oritur etiam ex solutione Problematis 19. §. 204.

Tertio si fiat $x=e^z$ sumpta pro e basi logarithmica hyperbolica habebitur $lx=z$; $dx=e^z dz$; $\int\frac{dx}{(lx)^n}=\int\frac{e^z dz}{z^n}=$

$$\int\frac{dz}{z^n}\left(1+z+\frac{z^2}{2}+\frac{z^3}{2.3}+\&c.\ldots\right)=G+\frac{z^{1-n}}{1-n}+\frac{z^{2-n}}{2-n}+\frac{z^{3-n}}{2(3-n)}+\frac{z^{4-n}}{2.3(4-n)}+\frac{z^{5-n}}{2.3.4(5-n)}+\&c.$$ ac tandem

III°. $y=G-\frac{1}{(n-1)(lx)^{n-1}}-\frac{1}{(n-2)(lx)^{n-2}}-\frac{1}{2(n-3)(lx)^{n-3}}-\frac{1}{2.3(n-4)(lx)^{n-4}}-\&c.$

quae series quoties n est numerus integer positivus habet specie tenus unum terminum valoris infiniti; sed revera loco illius termini recipit integrale logarithmicum. Omissis itaque casibus, in quibus n est numerus integer sive positivus, sive negativus, qui satis adhuc explicati sunt; possent considerari hae tres integrationes pro casibus, in quibus n est numerus fractus, aut quicumque irrationalis. Sed non erit difficile ex iis quae supra explicavimus has quaestiones absolvere.

ADDI-

ADDITAMENTUM.

Iam ſuperiora praelum ſubierant, cum celeber. V. D. Gregorius Fontana mihi ſequentia perhumanis litteris exhibuit.

„ In meis commentariis reperio ratiocinium Euleri circa „ notam formulam $\int \frac{dz}{\log. z}$ " (quo nempe ratiocinio conſtituitur poſito integrali aequali zero pro caſu $z = o$, fieri ipſum integrale infinitum pro caſu $z = 1$) „ poſſe confirmari „ hoc modo: in ſubſtitutione $z = 1 - \omega$ non debet conſiderari „ ω ſemper infiniteſima, ſed talis ut dum z creſcit a „ zero uſque ad unitatem; ω decreſcat ab unitate ad zero. „ Revera cum ſit $z = 1 - \omega$, erit $\frac{dz}{z} = \frac{-d\omega}{1-\omega}$, & $\log. z = \log.$ „ $(1 - \omega) = -\omega - \frac{1}{2}\omega^2 - \frac{1}{3}\omega^3 - \frac{1}{4}\omega^4 - \&c.$ ſine „ conſtanti, nam prima conditio ſervatur. Hoc poſito erit

$$\frac{dz}{\log. z} = \frac{d\omega}{\omega + \frac{1}{2}\omega^2 + \frac{1}{3}\omega^3 + \frac{1}{4}\omega^4 + \&c.} =$$

$$„ \frac{d\omega}{\omega} - \frac{d\omega}{2} - \frac{\omega d\omega}{3.4} - \frac{\omega^2 d\omega}{2.3.4} - \frac{19\omega^3 d\omega}{2.3.4.5.6} - \&c.$$

„ ergo integrando adhibita neceſſaria conſtanti habebimus

$$„ \int \frac{dz}{\log. z} = \log. \omega + \frac{1-\omega}{2} + \frac{1-\omega^2}{2.3.4} + \frac{1-\omega^3}{3.2.3.4} + \frac{19(1-\omega^4)}{4.2.3.4.5.6} + \&c.$$

„ Hoc modo ſervatur conditio, quod facto $z = o$, ſeu $\omega = 1$ „ ſit ipſum integrale $= o$. Ergo facto $z = 1$, ſeu $\omega = o$, „ obtinebitur

$$„ \int \frac{dz}{\log. z} = \log. o + \frac{1}{2} + \frac{1}{2.3.4} + \frac{1}{3.2.3.4} + \frac{19}{4.2.3.4.5.6} + \&c.$$

„ Ut nunc habeatur valor ſeriei $\frac{1}{2} + \frac{1}{2.3.4} + \frac{1}{3.2.3.4} +$

" $\frac{19}{4.2.3.4.5.6}$ + &c. observo esse $1 - \frac{1}{1+\frac{x}{2}+\frac{x^2}{3}+\frac{x^3}{4}+\frac{x^4}{5}+\&c.}$
" $= \frac{x}{2} + \frac{x^2}{3.4} + \frac{x^3}{2.3.4} + \frac{19x^4}{2.3.4.5.6}$ + &c., ut patet ex redu-
" ctione fractionis in seriem. Itaque facto $x = 1$; erit
" $1 - \frac{1}{1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\&c.} = \frac{1}{2} + \frac{1}{3.4} + \frac{1}{2.3.4} + \frac{19}{2.3.4.5.6}$ + &c.
" Sed fractio $\frac{1}{1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\&c.}$ est aequalis zero ratione
" denominatoris infiniti. Ergo resultat
" $1 = \frac{1}{2} + \frac{1}{3.4} + \frac{1}{2.3.4} + \frac{19}{2.3.4.5.6}$ + &c., ac proinde
" $\frac{1}{2} + \frac{1}{2.3.4} + \frac{1}{3.2.3.4} + \frac{19}{4.2.3.4.5.6} + \&c. < 1$.
" Itaque tandem habebimus $\int \frac{dz}{\log. z} = \log. o$ + quantitate minore
" quam sit unitas sive $\int \frac{dz}{\log. z}$ = infinito negativo quando $z = 1$ ".

Superior aequatio Fontanae

$$(a) \ldots \int \frac{dz}{lz} = l\omega + \frac{1-\omega}{2} + \frac{1-\omega^2}{2.3.4} + \frac{1-\omega^3}{3.2.3.4} + \frac{19(1-\omega^4)}{4.2.3.4.5.6} \&c. \ldots$$

generalis est pro quocumque valore $z = 1 - \omega$ inter o, & 1, aequatio vero superius a nobis posita

$$\int \frac{dz}{lz} = A + l - lz + lz + \frac{(lz)^2}{2.2} + \frac{(lz)^3}{2.3.3} + \&c.$$

posito $1 - \omega$ loco z abit in sequentem

$$\int \frac{dz}{lz} = A + l\left(\omega + \frac{\omega^2}{2} + \frac{\omega^3}{3} + \&c.\right) - \omega - \frac{\omega^2}{2} - \&c.$$
$$+ \frac{\omega^2}{2.2} + \&c.$$
$$- \&c.$$

Est

Est autem $l\left(\omega+\frac{\omega^2}{2}+\frac{\omega^3}{3}+\&c.\right)=l\omega+S$, existente S serie terminorum qui afficiuntur potentiis ipsius ω. Erit ergo $(6)\ldots\int\frac{dz}{lz}=A+l\omega+\Omega$ collectis in Ω terminis omnibus qui afficiuntur potentis ipsius ω. Cum ergo Ω identificari debeat cum serie Fontanae $-\frac{\omega}{2}-\frac{\omega^2}{2.3.4}-\frac{\omega^3}{3.2.3.4}-\&c.$ & cum in utraque aequatione (a), & (6) insit terminus $l\omega$; erit constans A aequalis constanti Fontanae, quam ipse demonstravit esse unitate minorem. Scilicet erit

$$A=0,\ 577215\ 664901\ 532860\ 618112\ 090082\ 39$$
$$=\frac{1}{2}+\frac{1}{2.3.4}+\frac{1}{3.2.3.4}+\frac{19}{4.2.3.4.5.6}+\&c.\ldots,$$

cuius seriei quatuor priores termini actu collecti dant numerum 0, 562152

Explicatio necessitatis signi duplicis ± adhibendi in integratione logarithmica.

Supponimus hoc loco doctrinam Euleri, cum quo plures Mathematici consentiunt, inter quos Fontana in Monum. Soc. Ital. Vol. I., omnes logarithmos quantitatis negativae esse imaginarios. Etenim apud eos qui putant logarithmum quantitatis affectae signo negativo esse eundem cum logarithmo eiusdem quantitatis affectae signo positivo, inter quos quoque numerantur Mathematici summi nominis; nulla erit necessitas, aut utilitas signi ± adhibiti post signum logarithmicum, cum tam signum + quam signum — idem praestet quo ad valorem logarithmi.

Supposito itaque quod logarithmi quantitatum negativarum sint imaginarii; quotiescumque habetur differentiale logarithmi-

rithmicum reale, in cuius integratione quantitas, quae cadit sub signo logarithmico per variationem variabilis ipsam ingredientis potest fieri negativa, & quidem per talem variationem variabilis, quae non efficiat ut differentiale ipsum desinat esse reale, tunc quaecumque quantitas constans addatur in integratione; semper in integrali logarithmico post signum logarithmicum ante quantitatem, quae per variationem variabilis intra datam conditionem fieri potest negativa, poni debet signum duplex $\pm$. Huius autem signi duplicis pars alteri contraria tunc sumi incipiet, cum valor quantitatis signo duplici affectae a negativo transit in positivum, aut viceversa. Huius doctrinae pars prior quod nempe signum duplex $\pm$ ponendum sit post signum logarithmicum in integratione logarithmica probatur ex eo quod est $\frac{dx}{x}=\frac{-dx}{-x}$; ex quo licet nos non inferamus esse etiam $lx=l-x$, ut ii volunt qui statuunt eosdem esse logarithmos quantitatum positivarum ac negativarum; dicimus tamen, quod nemo negaverit, ipsum differentiale $\frac{dx}{x}$ tam oriri potuisse ex lx, quam ex $l-x$. Ergo aequum erit ut in integratione duplex illa origo indicetur; quare haberi debebit $\int\frac{dx}{x}=l\pm x$. Eodem modo quo quamvis ex $a^2=(-a)^2$ non possit inferri dividendo exponentem 2 per 2 esse $a=-a$; tamen habita aequatione $x^2=a^2$, ob $a^2=(-a)^2$ extracta radice scribendum erit $x=\pm a$. Secunda vero pars eiusdem doctrinae, quod nempe signi duplicis contrariae partes sint accipiendae altera loco alterius quoties quantitas signo duplici affecta per variationem variabilium ipsam ingredientium transit a valore positivo ad negativum aut contra, manente reali differentiali logarithmico inde confirmatur; quod manente reali differentiali logarithmico integrale non possit mutare naturam suam. Scilicet si

inte-

integrale iam erat imaginarium nempe ob conſtantem imaginariam additam functioni reali variabilis quae reſultat ex integratione; adhuc imaginarium manere debet quaecumque variatio acciderit ſuo differentiali dummodo ſemper reale permanſerit. Non poteſt enim integrale ex imaginario fieri reale niſi per additum imaginarium, quod partem imaginariam elidat, ac tollat. Hoc autem imaginarium addi aut tolli non poteſt per fluxionem perpetuo realem. Viceverſa ſi integrale iam erat reale; ſi nempe variabili reali addita fuit in integratione conſtans realis; tale ſemper remanere debebit quicumque ſit ſtatus ſui differentialis dummodo ſemper reale permanſerit. Hoc autem in integrationibus logarithmicis haberi non poteſt niſi adhibendo contrarias partes ſigni duplicis, ut praeceptum eſt. Quod ſi quantitas, quae per integrationem ponenda eſt ſub ſigno logarithmico talis ſit ut numquam per variationem variabilis tranſire poſſit a valore poſitivo ad negativum, aut viceverſa; tunc omittendum erit in integratione ſignum duplex.

Huius doctrinae licet nova exempla in ſequentibus occurrere debeant; tamen exemplum maxime perſpicuum, ac ſimplex non videtur hoc loco omittendum.

Eulerus ſequenti Cap. V. §. 248. invenit eſſe

$$\int \frac{d\varphi}{\text{ſin.}\,\varphi} = \frac{1}{2}\, l\, \frac{1-\text{coſ.}\,\varphi}{1+\text{coſ.}\,\varphi} = l\ \text{tang.}\ \frac{1}{2}\varphi$$

$$\int \frac{d\varphi}{\text{coſ.}\,\varphi} = \frac{1}{2}\, l\, \frac{1+\text{ſin.}\,\varphi}{1-\text{ſin.}\,\varphi} = l\ \text{tang.}\left(45^\circ + \frac{1}{2}\varphi\right)$$

Nunc ſi in huiuſmodi aequationibus accipiantur quantitates poſitae ſub ſigno logarithmico prout iacent; erit pro pluribus valoribus ipſius φ ſemilogarithmus quantitatis poſitivae aequalis logarithmo negativae, contra ſuppoſitionem. Sit enim $\frac{1}{2}\varphi = q\pi - r\frac{\pi}{2}$, ubi q eſt numerus poſitivus integer, r vero fractio poſitiva, ac π ſemiperipheria circuli cuius radius $= 1$.

 Erit

Erit pro his casibus tang. $\frac{1}{2}\varphi$ negativa. Est autem $\frac{1-\text{cos.}\,\varphi}{1+\text{cos.}\,\varphi}$ semper quantitas positiva. Ergo in priore aequatione erit dimidius logarithmus quantitatis positivae aequalis logarithmo negativae. Idem eveniet in secunda aequatione quoties fuerit $45^\circ + \frac{1}{2}\varphi = q\pi - r\frac{\pi}{2}$.

Posito ergo quod absurdum credamus quantitatem realem aequari posse logarithmo quantitatis negativae; scribendum erit

$$\int \frac{d\varphi}{\text{sin.}\,\varphi} = \frac{1}{2} l \frac{1-\text{cos.}\,\varphi}{1+\text{cos.}\,\varphi} = l \pm \text{tang.}\,\frac{1}{2}\varphi$$

$$\int \frac{d\varphi}{\text{cos.}\,\varphi} = \frac{1}{2} l \frac{1+\text{sin.}\,\varphi}{1-\text{sin.}\,\varphi} = l \pm \text{tang.}\left(45^\circ + \frac{1}{2}\varphi\right)$$

posito nempe signo $\pm$ post signum logarithmicum dumtaxat ante eas quantitates, quae per variationem variabilis transire possunt a valore positivo ad negativum; cuius signi pars superior $+$ adhibenda erit pro valoribus positivis tangentis cui praeponitur; pars vero inferior $-$ pro valoribus negativis eiusdem; adeo ut sub signo logarithmico ex utraque parte aequationis semper valores positivi reperiantur.

Idem vero etiam aliunde confirmatur hoc modo. Cum sit

$$\left(\text{tang.}\,\frac{1}{2}\varphi\right)^2 = \frac{(\text{sin.}\,\frac{1}{2}\varphi)^2}{(\text{cos.}\,\frac{1}{2}\varphi)^2} = \frac{1-(\text{cos.}\,\frac{1}{2}\varphi)^2+(\text{sin.}\,\frac{1}{2}\varphi)^2}{1+(\text{cos.}\,\frac{1}{2}\varphi)^2-(\text{sin.}\,\frac{1}{2}\varphi)^2}$$

$$= \frac{1-\text{cos.}(\frac{1}{2}\varphi+\frac{1}{2}\varphi)}{1+\text{cos.}(\frac{1}{2}\varphi+\frac{1}{2}\varphi)} = \frac{1-\text{cos.}\,\varphi}{1+\text{cos.}\,\varphi};\ \text{erit}$$

$$\text{tang.}\,\frac{1}{2}\varphi = \mp\sqrt{\frac{1-\text{cos.}\,\varphi}{1+\text{cos.}\,\varphi}};\ \text{seu} \pm \text{tang.}\,\frac{1}{2}\varphi = \sqrt{\frac{1-\text{cos.}\,\varphi}{1+\text{cos.}\,\varphi}},$$

ubi $\sqrt{\frac{1-\text{cos.}\,\varphi}{1+\text{cos.}\,\varphi}}$ accipitur semper positive. Evidens autem est posita hac conditione signum $\pm$ praeponendum esse ante

tang.

tang. $\frac{1}{2}\varphi$ iuxta regulas algebrae, in eoque ut vera sit aequatio sumendum esse signum + cum tangens $\frac{1}{2}\varphi$ est positiva; signum vero — cum eadem tang. $\frac{1}{2}\varphi$ sit negativa per variationem ipsius φ. Ergo praeposito utrinque signo logarithmico habebitur iisdem conditionibus

$$l.\sqrt{\frac{1-\text{cos.}\,\varphi}{1+\text{cos.}\,\varphi}} = \frac{1}{2}\,l.\frac{1-\text{cos.}\,\varphi}{1+\text{cos.}\,\varphi} = l \pm \text{tang.}\frac{1}{2}\,\varphi.$$

Eodem modo demonstratio instituitur pro secunda aequatione.

Hoc exemplum eo opportunius est, quod in illo apparet nexus regulae signi duplicis $\pm$ adhibendi in extractione radicis quadratae cum regula eiusdem signi adhibendi in integratione logarithmica.

Ut vero in hoc exemplo nihil omittamus, quod ad rem nostram facere possit; praestabit instituere integrationem formulae $\frac{d\varphi}{\text{sin.}\,\varphi}$ etiam per seriem infinitam. Sit sin. $\varphi = u$; erit

$$\varphi = \text{Arc. sin. } u\ ;\ d\varphi = \frac{du}{\sqrt{(1-u^2)}};\ \frac{d\varphi}{\text{sin.}\,\varphi} =$$

$$\frac{du}{u\sqrt{(1-u^2)}} = \frac{du}{u}\left(1+\frac{1}{2}u^2+\frac{1.3}{2.4}u^4+\frac{1.3.5}{2.4.6}u^6+\ldots\right)$$

$$\int\frac{d\varphi}{\text{sin.}\,\varphi} = \text{Const.} + lu + \frac{u^2}{2.2} + \frac{1.3\,u^4}{2.4.4} + \frac{1.3.5\,u^6}{2.4.6.6} + \&c\ldots$$

$$= \text{Const.} + l\,\text{sin.}\,\varphi + \frac{(\text{sin.}\,\varphi)^2}{2.2} + \frac{3(\text{sin.}\,\varphi)^4}{2.4.4} + \frac{3.5\,(\text{sin.}\,\varphi)^6}{2.4.6.6} +$$

Quae constans si determinetur ut integrale evanescat quando $\varphi = \frac{\pi}{2}$ uti supra factum est; erit

$\int\frac{d\varphi}{\text{sin.}\varphi}$

$$\int \frac{d\varphi}{\text{fin.}\varphi} = \frac{1}{2} l \frac{1-\text{cof.}\varphi}{1+\text{cof.}\varphi} = l\,\text{fin.}\varphi + \frac{1-(\text{fin.}\varphi)^2}{2.\,2} + \frac{3(1-(\text{fin.}\varphi)^4)}{2.\,4.\,4} + \ldots$$

ubi nifi fcribatur fignum $\pm$ in $l.$ fin. φ; cum eft $\varphi = 2q\pi - r\pi$ exiftente q numero integro pofitivo, r vero fractione, atque adeo cum fin. φ eft negativus; quantitas realis aequaretur quantitati mixtae ex realibus, & logarithmo quantitatis negativae. Si vero addatur fignum $\pm$; atque ita adhibeatur, ut $l \pm$ fin. φ. fit femper logarithmus quantitatis pofitivae; quemadmodum non variatur valor quantitatis $\frac{1}{2} l \frac{1-\text{cof.}\varphi}{1+\text{cof.}\varphi}$ quando a in φ loco valoris pofitivi fumitur idem valor negative; ita neque valor feriei, cui aequatur.

Oritur tamen hic alia difficultas, quod idem fit fin. φ quando $\varphi = q\pi + r\frac{\pi}{2}$, & quando $\varphi = (q+1)\pi - r\frac{\pi}{2}$ non folum quo ad qualitatem, fed etiam quo ad quantitatem valoris pofitivi, aut negativi. Si vero cof. $(q\pi + r\frac{\pi}{2})$ fit pofitivus, erit negativus eiufdem quantitatis cof. $\left((q+1)\pi - r\frac{\pi}{2}\right)$.

Quare variabitur expreffio $\frac{1}{2} l \frac{1-\text{cof.}\varphi}{1+\text{cof.}\varphi}$, quin varietur ullo modo eius valor

$$l \pm \text{fin.}\varphi + \frac{1-(\text{fin.}\varphi)^2}{2.\,2.} + \frac{3(1-(\text{fin.}\varphi)^4)}{2.\,4.\,4.} + \ldots.$$

Huic incommodo occurritur confiderando, quod fi arcui $q\pi + r\frac{\pi}{2}$ fubftituatur arcus $(q+1)\pi - r\frac{\pi}{2}$; expreffio $\frac{1}{2} l \frac{1-\text{cof.}\varphi}{1+\text{cof.}\varphi}$ tranfit a pofitiva in negativam, aut contra, quin quantitas illius mutetur. Ex alia parte cum affumpta fuerit

fuerit aequatio $d\varphi = \frac{du}{\sqrt{(1-u^2)}}$, quae revera est

$d\varphi = \frac{du}{\pm\sqrt{(1-u^2)}}$, ut attendenti ipsos duos casus

$\varphi = q\pi + r\frac{\pi}{2}$, & $\varphi = (q+1)\pi - r\frac{\pi}{2}$ perspicuum est;

inde fiet $\int\frac{d\varphi}{\sin.\varphi} = \pm\int\frac{du}{u\sqrt{(1-u^2)}}$; ac proinde

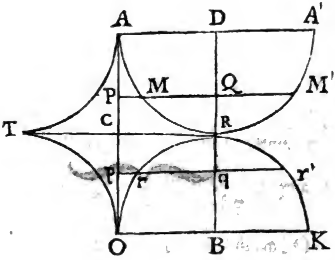
$$\int\frac{d\varphi}{\sin.\varphi} = \frac{1}{2}l\frac{1-\cos.\varphi}{1+\cos.\varphi} = \pm\left[l\pm\sin.\varphi + \frac{1-(\sin.\varphi)^2}{2.\,2} + \frac{3(1-(\sin.\varphi)^4)}{2.\,4.\,4} + ..\right]$$

In qua aequatione omnia praecaventur, quae fuerant praecavenda.

Solutio

Solutio cuiusdam paradoxi propositi ab Alembertio per signum $\pm$ rite adhibitum in integratione:

Quamquam integratio, quam examinabimus non sit logarithmica; tamen quia sine paradoxo expeditur per signum $\pm$ convenienter applicatum; visum est eam hic per occasionem collocare, quando iam satis huius signi necessitatem, ac regulas constituimus pro integratione logarithmica, neque pro aliis capitibus Calculi Integralis Auctoris nostri fiet opportunum explicare regulas signi $\pm$ in integratione Alembertiana, aliisque similibus adhibendi.



Alembertius Tom. 4. Opusc. pag. 65. postquam amico nonnulla alia paradoxa proposuerit; haec habet: „ En aliam „ speciem paradoxi, de quo iam egi in Vol. 3. Monum. „ Berolin. anni 1747., quin solutionem invenerim, quae „ mihi satis placuerit. Sit PM $=y$ (Fig. 1.); AP $=x$; „ $dy = dx\sqrt{((1-x)^{-\frac{2}{3}}-1)}$, (Monum. Berol. 1747. p. 241.); AC $=1$.

„ AC = 1. Elementum arcus AM est $dx\sqrt{(1-x)^{-\frac{2}{3}}}$,
„ cuius integrale est $\int dx(1-x)^{-\frac{1}{3}}$, sive $-\frac{3}{2}(1-x)^{\frac{2}{3}}+\frac{3}{2}$,
„ vel facto $1-x=z=CP$, $\frac{3}{2}\,(1-CP^{\frac{2}{3}})$. Si $CP=0$, ha-
„ betur $AR=\frac{3}{2}$. Si CP sit negativum habebitur valor
„ $ARr=\frac{3}{2}(1-(-CP)^{\frac{2}{3}})$, qui ob $(-CP)^2=CP^2$ est
„ idem cum $\frac{3}{2}(1-CP^{\frac{2}{3}})$; quod tamen secus esse debet,
„ cum sit $ARr > AM$ posito $Cp=CP$. En igitur etiam hoc
„ loco deficientem calculum, quandoquidem ut plenius satis-
„ fiat aequationi $dy=dx\sqrt{((1-x)^{-\frac{2}{3}}-1)}$ sumpto radicali
„ positivo, supponi debet quod curva, quae prosequitur ultra
„ punctum R iam non sit RO, sed RK aequalis ac similis
„ ipsi RO “.

Iniuria tamen accusatur calculus. Quod ut clarius demonstremus nonnulla sunt praemittenda. Sumpto radicali positivo in aequatione $dy = dx\sqrt{((-x)^{-\frac{2}{3}}-1)}$, seu

$$dy = dx\,(1-x^{-\frac{2}{3}}-1)^{\frac{1}{2}} = -\,dz\,(z^{-\frac{2}{3}}-1)^{\frac{1}{2}},$$

$= -z^{-\frac{1}{3}}dz(1-z^{\frac{2}{3}})^{\frac{1}{2}}$; cum sit

$$(1-z^{\frac{2}{3}})^{\frac{1}{2}} = 1-\frac{1}{2}z^{\frac{2}{3}}-\frac{1}{2.4}z^{\frac{4}{3}}-\frac{3}{2.4.6}z^{\frac{6}{3}}-\frac{3.5}{2.4.6.8}z^{\frac{8}{3}}-\ldots$$

habebitur

$$y = B-\frac{3}{2}z^{\frac{2}{3}}+\frac{1}{2}.\frac{3}{4}z^{\frac{4}{3}}+\frac{1}{2.4}.\frac{3}{6}z^{\frac{6}{3}}+\frac{3}{2.4.6}.\frac{3}{8}z^{\frac{8}{3}}+\ldots(1)$$

unde posito quod y annihiletur quando $z=1$ iuxta suppositionem Alembertii; erit

B =

$$B = \frac{3}{2} - \frac{1}{2} \cdot \frac{3}{4} - \frac{1}{2.4} \cdot \frac{3}{6} - \frac{3}{2.4.6} \cdot \frac{3}{8} - \ldots\ldots$$ Quare cum sit $\frac{1}{2} + \frac{1}{2.4} + \frac{3}{2.4.6} + \frac{3.5}{2.4.6.8} + \&c. \ldots\ldots = 1$

ut resultat ex evolutione $(1 - 1)^{\frac{1}{2}} = 0$; erit

$$\frac{1}{2} \cdot \frac{3}{4} + \frac{1}{2.4} \cdot \frac{3}{6} + \frac{3}{2.4.6} \cdot \frac{3}{8} + \ldots\ldots < 1$$

ac proinde B quantitas positiva $= CR$, qui est valor ipsius y quando $z = 0$.

Ex hoc calculo in primis resultat quod sumpto radicali positivo eadem ordinata y respondet tam valori positivo z quam negativo. Quare sumpto radicali positivo; curva quae prosequitur ultra punctum R erit revera RO.

Ut indoles tota huius curvae melius appareat, sumatur nunc recta DRB parallela ipsi AC pro axe abscissarum $z = RQ = CP$, ac RC normalis ipsi DB, sumatur pro axe ordinatarum $u = QM = CR - PM = B - y$; habebitur aequatio $-dy = du = dz\,(z^{-\frac{2}{3}} - 1)^{\frac{1}{2}}$; ac proinde

$$u = \frac{3}{2} z^{\frac{2}{3}} - \frac{1}{2} \cdot \frac{3}{4} z^{\frac{4}{3}} - \frac{1}{2.4} \cdot \frac{3}{6} z^{\frac{6}{3}} - \frac{3}{2.4.6} \cdot \frac{3}{8} z^{\frac{8}{3}} - \ldots$$

sine additione constantis ut simul evanescat u & z in R.

Cum vero in aequatione $du = dz \sqrt{(z^{-\frac{2}{3}} - 1)}$ contineatur ratione signi $\sqrt{}$ duplex valoris species adeo ut sit $du = \pm\, dz\,(z^{-\frac{2}{3}} - 1)^{\frac{1}{2}}$; habebitur revera generaliter

$$u = \pm\left(\frac{3}{2} z^{\frac{2}{3}} - \frac{1}{2} \cdot \frac{3}{4} z^{\frac{4}{3}} - \frac{3}{2.4.6} \cdot \frac{3}{6} z^{\frac{6}{3}} - \frac{3.5}{2.4.6.8} \cdot \frac{3}{8} z^{\frac{8}{3}} + \ldots\right) (2).$$

Atque haec est aequatio maxime propria ad perspiciendam naturam curvae.

Ex hac aequatione primo intelligitur curvam habere quatuor ramos similes, & aequales RA, RO, RA', RK, ac

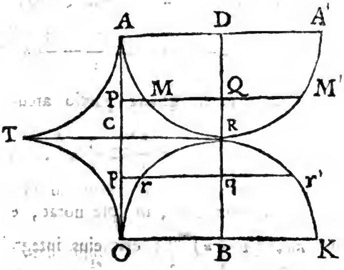
ac praeterea nullos. Quare perperam positi sunt ab Alembertio rami AT, TO.

Secundo: hos ramos terminari ex abrupto in quatuor punctis A, O, K, A' cum pro valore $z > 1$ fiat du imaginarium.

Revera etiam pro integratione pro axe Alembertii formulae $dy = \pm dx \sqrt{((1-x)^{-\frac{2}{3}} - 1)} = \mp dz (z^{-\frac{2}{3}} - 1)^{\frac{1}{2}}$ generaliter sumptae ope signi $\pm$ habetur

$$y = B \pm \left(-\frac{3}{2} z^{\frac{2}{3}} + \frac{1}{2} \cdot \frac{3}{4} z^{\frac{4}{3}} + \frac{1}{2.4} \cdot \frac{3}{6} z^{\frac{6}{3}} + \ldots\right) \qquad (3)$$

ubi constans $B = CR$ non debet affici signo $\pm$. Quare pro aliquo valore z puta RQ habebitur alter valor $y = PM = PQ - QM$; alter vero $y = PM' = PQ + QM'$. Quod idem habetur ex aequatione $y = B - u$ prout u accipitur positivum, aut negativum.



Nunc pro rectificatione, curvae cuius aequatio $du = \pm dz (z^{-\frac{2}{3}} - 1)^{\frac{1}{2}}$ habetur $du^2 + dz^2 = dz^2 . z^{-\frac{2}{3}}$;

atque elementum curvae $\surd(du^2 + dz^2)$ generaliter sumptum erit $\pm dz . z^{-\frac{1}{3}}$; ac proinde integrale erit $\pm \frac{3}{2} z^{\frac{2}{3}}$, quod annihilatur simul cum u, ac z. Erit ergo arcus in genere $= \pm \frac{3}{2} z^{\frac{2}{3}}$.

Paradoxum videri posset quod hic arcus adhuc exprimatur per formulam valoris realis etiam quando $z > 1$, in quo casu debet esse imaginarius ob ordinatam curvae imaginariam, in quod paradoxum incidit etiam positio axis Alembertii; quamvis de hoc nihil adnotaverit. Sed de hoc paradoxo, quod apparet in plurimis aliis curvis, agetur in sequenti paragrapho.

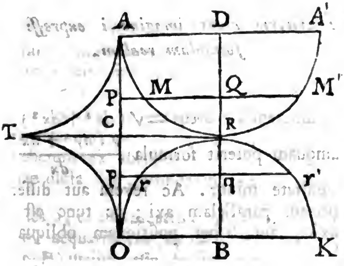
Si nunc sumatur arcus RMA positivus originem habens in R, eumque continuare libeat cum arcu Rr′K; habebimus directionem Rr′ negativam. Itaque si sumatur RQ = Rq, ac sit $RM = + \frac{3}{2} z^{\frac{2}{3}}$; erit $Rr' = - \frac{3}{2} z^{\frac{2}{3}}$, quod iam est evidens. Erit ergo $RA = \frac{3}{2}$; $AM = \frac{3}{2} - \frac{3}{2} z^{\frac{2}{3}}$;

$Ar' = \frac{3}{2} + \frac{3}{2} z^{\frac{2}{3}}$; ac in genere portio arcus ARK, cuius initium statuitur in A erit $= \frac{3}{2} \mp \frac{3}{2} z^{\frac{2}{3}}$. Idem resultat etiam ex integratione Alembertii dummodo in ipsa rite adhibeatur signum $\pm$. Nam cum, ut ipse notat, elementum arcus AM sit $dx \surd (1 - x)^{-\frac{2}{3}}$; erit eius integrale $\pm \int dx (1 - x)^{-\frac{1}{3}} = \frac{3}{2} \mp \frac{3}{2} (1 -)^{\frac{2}{3}}$ cum signum $\mp$ non afficiat constantem $\frac{3}{2}$. Quod non debeat sumi $+ \int dx (1 - x)^{-\frac{1}{3}}$

tam

tam pro arcu AM, quam pro arcu Ar', ut ſumpſit Alembertius, exinde patet quod differentiale in M, & r' debeat habere ſigna contraria pro varia ſpecie valorum arcuum RM, & Rr'.

Neque abſurdum videri debet quod differentiale quantitatis ſemper creſcentis ſit quandoque negativum quandoque poſitivum. Nam ſit $a - z$ quantitas ſemper creſcens a zero uſque in infinitum; erit eius differentiale $-dz$ modo negativum modum poſitivum prout erit poſitiva vel negativa quantitas z. Quod rite eſt animadvertendum in differentialibus quantitatum, quae per integrationem acquirunt conſtantem; cuiuſmodi ſunt differentialia ipſa Alembertii $dx\sqrt{((1-x)^{-\frac{2}{3}}-1)}$, ac $dx\sqrt{(1-x)^{-\frac{2}{3}}}$, quorum integralia debent annihilari quando $x=0$.



Quod vero ſignum $\pm$ praepoſitum differentiali non debeat afficere conſtantem, quae ingreditur per integrationem, ſed dumtaxat partem variabilem ipſius integralis, quae immediate oritur ex differentiali; quamvis iam demonſtratum ſit; tamen etiam hoc noto exemplo illuſtrari poteſt. Sit curvae

AMR aequatio $dy = \frac{x\,dx}{\sqrt{(1-xx)}}$; posito AP $= x$, PM $= y$; erit $y = C - \sqrt{(1-xx)}$; ubi si y evanescere debeat simul cum x; habebitur $C = 1$, quare $y = 1 - \sqrt{(1-xx)}$; est autem haec aequatio quadrantis circuli, & generaliter $y = 1 \pm \sqrt{(1-xx)}$ aequatio semicirculi ARA′.

Cum haec curva non redeat in seipsam, non potest inferri ex rectificabilitate eius arcuum argumentum contra demonstrationem Newtoni, quod nullae curvae in se redeuntes sint rectificabiles.

Hoc vero genus curvarum, quae in se non redeunt, neque abeunt in infinitum, sed terminantur ex abrupto non videtur explicatum fuisse a Geometris. Non est autem difficile infinitas curvas huius generis invenire, quod mox docebimus.

De criterio arcus imaginarii expressi per formulam realem.

CUm sit differentiale arcus $= \sqrt{(dy^2 + dx^2)}$ existente reali x, & y, numquam poterit formula $\frac{\sqrt{(dy^2+dx^2)}}{dx}$ positive sumpta esse unitate minor. Ac revera aut differentiale arcus habet positionem parallelam axi; ac tunc est aequale differentiali axis; aut habet positionem obliquam, ac tunc est maius.

Hoc posito si sit arcus $s = X$, quae sit functio ipsius x, ac sit $dX = P\,dx$, sit vero P quantitas, quae pro aliquo valore ipsius x fiat fracta; pro eo valore x licet sit X quantitas realis; arcus tamen s erit imaginarius.

Ratio est quod expressio s, quae nihil aliud est quam Arc. absc. x, includit conditiones geometricas, quae pro valoribus

loribus nonnullis licet realibus ipsius X locum habere non possunt.

Infinitae ergo curvae esse possunt, in quibus arcus imaginarii mentiantur valorem realem, cum infinitae sint formulae X, in quarum differentialibus Pdx functio P pro aliquo valore x fiat quantitas fracta.

Facto ut supra $s = X$; $ds = \sqrt{(dy^2 + dx^2)} = Pdx$; habebitur $dy^2 = dx^2 (P - 1)$; $dy = dx\sqrt{(P - 1)}$. Quando P sit quantitas fracta, tunc dy evadit imaginarium. Ergo limes, in quo arcus imaginarius incipit mentiri valorem realem, est idem limes, in quo ordinata cum suo differentiali sit imaginaria, atque item exprimitur per formulam imaginariam.

Si in integratione formulae $dy = dx\sqrt{(P - 1)}$ nulla addatur constans; in curva quae inde enascetur relata ad axes normales x, & y pro quocumque valore determinato x, non poterit y habere nisi duos valores aequales affectos signis contrariis, ac proinde si hi valores sint finiti quando $P = 1$; curva habebit ramos duos deficientes ex abrupto, si paullisper immutato valore x iam fiat $P < 1$.

Quot erunt ergo valores x, quibus respondeat aequatio $P = 1$, adeo ut paullisper immutato valore x, fiat $P < 1$, tot erunt paria aequalia, & similia ramorum curvae, qui pro iis valoribus deficiunt ex abrupto.

Ex iis, quae dicta sunt, pronum est iudicare de curva Alembertii supra explicata, cuius aequatio est $dy = dx\sqrt{((1-x)^{-\frac{2}{3}} - 1)}$, quae refertur ad aequationem $dy = dx\sqrt{(P - 1)}$, ubi pro casu Alembertii P incipit esse fractio quando x incipit esse negativum, aut positivum > 2.

Ex his curvis infinitis quasi prima est, quae exprimitur aequatione $dy = dx\sqrt{(a + bx)} = dx\sqrt{((a + 1 + bx) - 1)}$, quae simul est quadrabilis, ac rectificabilis. Si in hac aequatione sit $a = 0$ repraesentante abscissa x distantias planetarum a centro; ordinata y repraesentabit tempora periodica.

Adno-

Adnotatio II. ad Cap. V. Sect. I. Vol. I.

De integratione formularum $x^n dx \sin. x$, $x^n dx \cos. x$.

Cum de integratione huiusmodi formularum Eulerus nihil praecipiat, cumque satis conferant ad solutionem nobilium problematum, quae hactenus intacta fuerant; visum est earum tractationem addere Cap. V. Sectionis huius, cui nimirum capiti titulus est: *De integratione formularum angulos, sinusque angulorum implicantium*.

Constat autem ad formulas $x^n dx \sin. x$, $x^n dx \cos. x$ etiam has alias $z^m dz \sin.(z^c)$ $z^m dz \cos.(z^c)$ reduci posse posito $z^c = x$, unde habetur $z^m = x^{\frac{m}{c}}$; $dz = d.x^{\frac{1}{c}} = \frac{1}{c} x^{\frac{1-c}{c}} dx$;

$$z^m dz \sin.(z^c) = \frac{1}{c} x^{\frac{m+1-c}{c}} dx \sin. x;\; z^m dz \cos.(z^c) = \frac{1}{c} x^{\frac{m+1-c}{c}} dx \cos. x.$$

Problema I.

Formularum $x^n dx \sin. x$, $x^n dx \cos. x$ integrale invenire siquidem n denotet numerum integrum positivum.

Solutio.

Cum sit $\int x^n dx \sin. x = -x^n \cos. x + \int n x^{n-1} dx \cos. x$

$\int x^n dx \cos. x = \quad x^n \sin. x - \int n x^{n-1} dx \sin. x$

per opportunas substitutiones eruemus.

$$\int x^n dx \sin. x = -x^n \cos. x + n x^{n-1} \sin. x + n(n-1) x^{n-2} \cos. x - n(n-1)(n-2) x^{n-3} \sin. x - \&c\ldots \quad (A)$$

quae series constabit numero finito terminorum; habebimus nempe

$\int x dx$

$$\int x\,dx\,\text{ſin.}\,x = -x\,\text{coſ.}\,x + \text{ſin.}\,x$$
$$\int x^2\,dx\,\text{ſin.}\,x = -x^2\,\text{coſ.}\,x + 2x\,\text{ſin.}\,x + 2\,\text{coſ.}\,x - 2$$
$$\int x^3\,dx\,\text{ſin.}\,x = -x^3\,\text{coſ.}\,x + 3x^2\,\text{ſin.}\,x + 3.2\,x\,\text{coſ.}\,x - 3.2\,\text{ſin.}\,x$$
$$\int x^4\,dx\,\text{ſin.}\,x = -x^4\,\text{coſ.}\,x + 4x^3\,\text{ſin.}\,x + 4.3\,x^2\,\text{coſ.}\,x - 4.3.2\,x\,\text{ſin.}\,x - 4.3.2\,\text{coſ.}\,x + 4.3.2$$

&c. &c.

Eodemque modo habebimus

$$\int x^n\,dx\,\text{coſ.}\,x = x^n\,\text{ſin.}\,x + nx^{n-1}\,\text{coſ.}\,x - n(n-1)x^{n-2}\,\text{ſin.}\,x - n(n-1)(n-2)x^{n-3}\,\text{coſ.}\,x + \text{\&c.}\ldots \qquad (B)$$

ac proinde

$$\int x\,dx\,\text{coſ.}\,x = x\,\text{ſin.}\,x + \text{coſ.}\,x - 1$$
$$\int x^2\,dx\,\text{coſ.}\,x = x^2\,\text{ſin.}\,x + 2x\,\text{coſ.}\,x - 2\,\text{ſin.}\,x$$
$$\int x^3\,dx\,\text{coſ.}\,x = x^3\,\text{ſin.}\,x + 3x^2\,\text{coſ.}\,x - 3.2x\,\text{ſin.}\,x - 3.2\,\text{coſ.}\,x + 3.2$$
$$\int x^4\,dx\,\text{coſ.}\,x = x^4\,\text{ſin.}\,x + 4x^3\,\text{coſ.}\,x - 4.3x^2\,\text{ſin.}\,x - 4.3.2x\,\text{coſ.}\,x + 4.3.2\,\text{ſin.}\,x$$

&c. &c.

quae ita ſunt ſumpta, ut evaneſcant poſito $x = 0$

Scholion.

Duo termini generales $n(n-1)(n-2)\ldots(n-k)\,x^{n-k-1}\,dx\,\text{coſ.}\,x$, & $n(n-1)(n-2)\ldots(n-k)\,x^{n-k-1}\,dx\,\text{ſin.}\,x$ ſerierum (A), & (B) evolvuntur ex duobus ſummatoriis $\int n(n-1)(n-2)\ldots(n-k)\,x^{n-k-1}\,dx\,\text{ſin.}\,x$, & $\int n(n-1)(n-2)\ldots(n-k)\,x^{n-k-1}\,dx\,\text{coſ.}\,x$. Iam vero quando $k = n$, annihilatur ſimul cum ſuo differentiali integrale formulae $\int n(n-1)(n-2)\ldots(n-k)\,x^{n-k-1}\,dx\,\text{ſin.}\,x$. Licet autem in eodem caſu $k = n$ aequetur nihilo differentiale $n(n-1)(n-2)\ldots(n-k)\,x^{n-k-1}\,dx\,\text{coſ.}\,x$; tamen ſi inſtituatur eiuſdem integratio indicata per formulam ſummatoriam

$$\int n(n-1)(n-2)\ldots(n-k)\,x^{n-k-1}\,dx\left(1 - \frac{x^2}{2} + \frac{x^4}{2.3.4} - \ldots\right);$$

habetur pro integrali quantitas conſtans $n(n-1)(n-2)\ldots(n-(k-1))$, quae eſt ipſa adiicienda ſeriei (A), aut (B), ut evaneſcat poſito $x = 0$.

Proble-

Problema II.

Formularum $x^n dx \sin. x$, & $x^n dx \cos. x$ integrale investigare, siquidem n denotet numerum integrum negativum.

Solutio.

Cum sit $\int x^n dx \sin. x = \frac{1}{n+1} x^{n+1} \sin. x - \int \frac{1}{n+1} x^{n+1} dx \cos. x$

$$\int x^n dx \cos. x = \frac{1}{n+1} x^{n+1} \cos. x + \int \frac{1}{n+1} x^{n+1} dx \sin. x$$

patet neque hoc modo per substitutiones posse haberi seriem, quae constet numero finito terminorum, quae exhibeat valorem integralis quaesiti. Videamus ergo quaenam ex seriebus infinitis hunc valorem commodius exprimant. Cum sit

$$\int x^n dx \sin. x = \int x^n dx \left(x - \frac{x^3}{2.3} + \frac{x^5}{2.3.4.5} - \frac{x^7}{2.3.4.5.6.7} + ..\right) =$$

$$C + \frac{1}{n+2} x^{n+2} - \frac{1}{n+4} . \frac{x^{n+4}}{2.3} + \frac{1}{n+6} . \frac{x^{n+6}}{2.3.4.5} - \frac{1}{n+8} . \frac{x^{n+8}}{2.3..7} + ...$$

si n sit numerus negativus impar, atque si $\int x^n dx \sin. x$ annihiletur quando $x = 1$; habebitur facile per seriem convergentem valor constantis C.

Si vero n sit numerus par negativus; quo casu in serie apparet terminus infinitus $\frac{1}{0} . \frac{1}{2.3.4....(-1-n)}$; tunc illius loco, qui oritur ex vulgari integratione formulae $\int \frac{dx}{x} . \frac{1}{2.3.4..(-1-n)}$ substituatur terminus $\frac{1}{2.3.4...(-1-n)} lx$, qui annihilatur quando $x = 1$. Eodem modo cum sit

$\int x^2$

$$\int x^n dx \cos. x = \int x^n dx \left(1 - \frac{x^2}{2} + \frac{x^4}{2.3.4} - \frac{x^6}{2.3.4.5.6} + \ldots\right)$$

$$= C + \frac{1}{n+1} x^{n+1} - \frac{1}{n+3} . \frac{x^{n+3}}{2} + \frac{1}{n+5} . \frac{x^{n+5}}{2.3.4} - \ldots;$$

ſumpta ſerie ut iacet ſi n ſit numerus negativus par; ſi vero ſit impar, ſubſtituto loco infiniti termino logarithmico $\frac{1}{2.3.4\ldots.(-1-n)} lx$; habebitur facile conſtans C poſito quod $\int x^n dx \cos. x$ annihiletur quando $x = 1$.

Scholion 1.

Quando n eſt numerus quicumque negativus unitate maior; iam patet quomodo determinetur commode conſtans C annihilato integrali quando $x = 1$.

Scholion 2.

Quando n eſt numerus quicumque negativus unitate minor, ſeu fractus, aut poſitivus quicumque; annihilato integrali quando $x = 0$; invenitur ipſa $C = 0$.

Scholion 3.

Duplex hoc loco problema ſolutionem poſtulat; primum quaenam ſeries ſubſtituendae ſint ſuperioribus ad habendum valorem integralis quando x eſt numerus ſatis magnus, ac ſeries ab initio ſunt divergentes; ſecundum quinam ſit valor integralis quando $x = \infty$; hic enim ſaepiſſime invenitur finitus, ac maxime attendendus. Nos praecipua ſeligemus.

Problema III.

Determinare constantem A in aequatione

$$\int \frac{dx \cos . x}{x} = A + lx - \frac{x^2}{2.2} + \frac{x^4}{2.3.4.4} - \frac{x^6}{2.3.4.5.6.6} + \ldots [1]$$

posito quod integrale $\int \frac{dx \cos . x}{x}$ annihiletur quando $x = \infty$.

Solutio.

Cum sit $\int \frac{dx \cos . x}{x} = \frac{\sin . x}{x} - \frac{\cos . x}{x^2} - \int 2 \frac{dx \cos . x}{x^3}$

$= \frac{\sin . x}{x} - \frac{\cos . x}{x^2} - \int 2 \frac{dx}{x^3} \left(1 - \frac{x^2}{2} + \frac{x^4}{2.3.4} - \ldots \right) =$

$\frac{\sin . x}{x} - \frac{\cos . x}{x^2} + B + \frac{1}{x^2} + lx - \frac{x^2}{2.3.4} + \frac{x^4}{4.3.4.5.6} - \ldots [2]$,

ubi B est constans ingressa per integrationem, quae debet assumi talis, ut integrale evanescat posito $x = \infty$; positis in hac aequatione [2] loco $\frac{\sin . x}{x}$, & $\frac{\cos . x}{x^2}$ valoribus ortis ex evolutione functionum $\sin . x$ & $\cos . x$; habebuntur duo termini constantes 1, & $\frac{1}{2}$; caeteri afficientur potentiis ipsius x. Collata itaque aequatione [2] cum [1] seu lx cum lx, & terminis, qui afficiuntur potentiis ipsius x in una aequatione cum terminis correspondentibus iisdem in alia; debebit esse etiam constans A aequationis [1] aequalis terminis constantibus aequationis [2]. Erit itaque

$A = B + 1 + \frac{1}{2}$; $B = A - 1 - \frac{1}{2}$.

Assumatur nunc in genere

$\int$

$$\int \frac{dx \operatorname{cof.} x}{x} = \frac{\operatorname{fin.} x}{x} - \frac{\operatorname{cof.} x}{x^2} - 2\frac{\operatorname{fin.} x}{x^3} + 2.3\frac{\operatorname{cof.} x}{x^4} + 2.3.4\frac{\operatorname{fin.} x}{x^5}$$

$$- \ldots\ldots + 2.3.4\ldots\ldots(\mu - 1)\frac{\operatorname{cof.} x}{x^\mu}$$

$$+ \int 2.3.4\ldots\mu \frac{dx}{x^{\mu+1}}\left(1 + \frac{x^2}{2} + \frac{x^4}{2.3.4} - \ldots\ldots\right),$$

ubi μ eſt index terminorum ante formulam ſummatoriam, & quidem formae $4p$ exiſtente p numero integro. Evoluta formula ſummatoria habebitur

$$\int \frac{dx \operatorname{cof.} x}{x} = \frac{\operatorname{fin.} x}{x} - \frac{\operatorname{cof.} x}{x^2} - 2\frac{\operatorname{fin.} x}{x^3} + 2.3\frac{\operatorname{cof.} x}{x^4} + 2.3.4\frac{\operatorname{fin.} x}{x^5}$$

$$- \ldots\ldots + 2.3.4\ldots(\mu - 1)\frac{\operatorname{cof.} x}{x^\mu} + M$$

$$+ 2.3.4\ldots\ldots\mu\left[\frac{x^{-\mu}}{-\mu} - \frac{x^{2-\mu}}{2(2-\mu)} + \frac{x^{4-\mu}}{2.3.4(4-\mu)}\right.$$

$$\left. - \ldots\ldots + \frac{1}{2.3.4\ldots\ldots(\mu - 2)\,2\,x^2}\right] + lx$$

$$- \frac{x^2}{2(\mu+1)(\mu+2)} + \frac{x^4}{4(\mu+1)(\mu+2)(\mu+3)(\mu+4)} - \ldots$$

$\ldots\ldots$[3], ubi M eſt conſtans ingreſſa per integrationem eius conditionis, ut integrale annihiletur quando $x = \infty$. Habebitur etiam

$$\int \frac{dx \operatorname{cof.} x}{x} = \frac{\operatorname{fin.} x}{x} - \frac{\operatorname{cof.} x}{x^2} - 2\frac{\operatorname{fin.} x}{x^3} + 2.3\frac{\operatorname{cof.} x}{x^4} + 2.3.4\frac{\operatorname{fin.} x}{x^5}$$

$$- \ldots\ldots + 2.3.4\ldots(\mu - 1)\frac{\operatorname{cof.} x}{x^\mu} + 2.3.4\ldots\mu\frac{\operatorname{fin.} x}{x^{\mu+1}}$$

$$- 2.3.4\ldots(\mu + 1)\frac{\operatorname{cof.} x}{x^{\mu+2}}$$

$$- \int 2.3.4\ldots(\mu+2)\frac{dx}{x^{\mu+3}}\left(1 - \frac{x^2}{2} + \frac{x^4}{2.3.4}\ \ldots\right)\ldots[4]$$

 cuius

cuius secundi membri formula summatoria evoluta dabit $N + lx + S$, existente N constante notae conditionis, & S serie terminorum affectorum potentiis x. Positis autem in hac aequatione [4] loco terminorum $2.3.4\ldots.\mu\,\frac{\text{sin.}\,x}{x^{\mu+1}}$, & $2.3.4\ldots.(\mu+1)\frac{\text{cos.}\,x}{x^{\mu+2}}$ valoribus per series ortas ex evolutione sin. x, & cos. x; & inde eductis constantibus $\frac{1}{\mu+1}$, & $\frac{1}{\mu+2}$, atque additis ipsi constanti N; habebitur per superiora aequatio $M = N + \frac{1}{\mu+1} + \frac{1}{\mu+2}$; $N = M - \frac{1}{\mu+1} - \frac{1}{\mu+2}$, atque cum idem resultet etiam quando μ est formae $4p+2$; tandem concludetur

$$M = A - 1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \ldots\ldots - \frac{1}{\mu}$$

Sit nunc $\mu = x = \infty$; evanescent in aequatione [3] omnes termini ante M, tum priores ex sequentibus. Itaque sumendo terminos, qui remanent ad latera ipsius lx ad dexteram atque ad sinistram, confectisque duabus seriebus, habebitur

$$\int\frac{dx\,\text{cos.}\,x}{x} = A - 1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \ldots - \frac{1}{\mu} + lx$$

$$+\begin{cases}\dfrac{\mu(\mu-1)}{2x^2} - \dfrac{\mu(\mu-1)(\mu-2)(\mu-3)}{4x^4} + \ldots\ldots \\[2ex] -\dfrac{x^2}{2(\mu+1)(\mu+2)} + \dfrac{x^4}{4(\mu+1)(\mu+2)(\mu+3)(\mu+4)} - \ldots.\end{cases}$$

Quae duae series affectae potentiis x cum invicem destruantur ob terminos correspondentes aequales, & contrariis signis praeditos; erit $\int\frac{dx\,\text{cos.}\,x}{x} = A - 1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \ldots - \frac{1}{\mu} + lx$;

seu

seu ob $lx = l\mu = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{\mu} - \frac{1}{2} - \frac{A}{2}$
$+ \frac{B}{4} - \frac{C}{6} + \frac{D}{8} - \ldots$ (Adn. I.); erit $\int \frac{dx \cos. x}{x} = 0 =$
$A - \frac{1}{2} - \frac{A}{2} + \frac{B}{4} - \frac{C}{6} + \frac{D}{8} - \ldots$
$A = 0,\ 577215\ 664901\ 532860\ 618112\ 090082\ 39$

Scholion 1.

Si in aequatione $\int \frac{dx \cos. x}{x} = A + l\text{-}x - \frac{x^2}{2.2} + \frac{x^4}{2.3.4.4} - \ldots$ quantitas x assumpta fuisset negativa, atque conditio foret, ut integrale evanesceret quando $x = -\infty$; constans A eundem valorem esset sortita, ut facile calculum relegenti patet. Ex supra dictis etiam huiusmodi aequatio ita scribenda erit
$$\int \frac{dx \cos. x}{x} = A + l \pm x - \frac{x^2}{2.2} + \frac{x^4}{2.3.4.4} - \ldots$$
ubi signum quantitatis positae sub logarithmico ita debet accipi, ut logarithmus sit realis.

Scholion 2.

Habet igitur constans A eundem valorem in duabus aequationibus
$$\int \frac{dx\, e^x}{x} = A + l \pm x + x + \frac{x^2}{2.2} + \frac{x^3}{2.3.3} + \frac{x^4}{2.3.4.4} + \ldots$$
$$\int \frac{dx \cos. x}{x} = A + l \pm x - \frac{x^2}{2.2} + \frac{x^4}{2.3.4.4} - \frac{x^6}{2.3.4.5.6.6} + \ldots$$
posito quod utrumque integrale annihiletur quando $x = -\infty$.

Scholion 3.

Quaeri posset etiam methodo superius tradita valor constantis C in aequatione

$$\int \frac{dx \,\text{sin.}\, x}{x^2} = C + l \pm x - \frac{x^2}{2.2.3} + \frac{x^4}{4.2.3.4.5} - \&c. \ldots [5]$$

posito quod integrale annihiletur quando $x = \mp \infty$; verum cum sit $\int \frac{dx \,\text{sin.}\, x}{x^2} = - \frac{\text{sin.}\, x}{x} + \int \frac{dx \,\text{cos.}\, x}{x}$

$$= - \frac{\text{sin.}\, x}{x} + A + l \pm x - \frac{x^2}{2.2} + \frac{x^4}{2.3.4.4} - \&c. \ldots\ldots [6],$$

in qua aequatione constans A superius inventa satisfacit conditioni, ut integrale aequationis [6] annihiletur quando $x = \pm \infty$; satis erit comparare duos valores integralis $\int \frac{dx \,\text{sin.}\, x}{x^2}$ habitos ex aequationibus [5], & [6]. Educendo enim ex termino $- \frac{\text{sin.}\, x}{x} = - 1 + \frac{x^2}{2.3} - \frac{x^4}{2.3.4.5} + \ldots.$ constantem -1, atque eam addendo ipsi A; habebimus aequationem $C = A - 1$.

Scholion 4.

Sit nunc I valor, quem induit

$$\int \frac{dx \,\text{cos.}\, x}{x} = A + l \pm x - \frac{x^2}{2.2} + \ldots.$$ quando $x = 1$; sive sit

$$A - \frac{1}{2.2} + \frac{1}{2.3.4.4} - \frac{1}{2.3.4.5.6.6} + \ldots = I.$$ Posita aequatione

$$\int \frac{dx \,\text{cos.}\, x}{x} = A - I + l \pm x - \frac{x^2}{2.2} + \frac{x^4}{2.3.4.4} - \ldots\ldots$$

inte-

integrale annihilabitur quando $x = 1$; pro hac vero suppositione integrale erit $= -1$ quando $x = \pm \infty$.

Scholion 5.

Aequatio [3] facile praeparari potest ut exhibeat valorem integralis $\int \frac{dx \cos. x}{x}$, quando x est quantitas satis magna eodem modo quo in superiore Adnotatione praeparata fuit series (10). Quare in hoc argumento diutius non immorabimur. Illud unum advertemus quod si in formula $\int x^n dx \sin. x$ numerus n sit par negativus, aut si in formula $\int x^n dx \cos. x$ idem numerus sit impar negativus; in iis casibus per aequationes Problematis II.

$$\int x^n dx \sin. x = \frac{1}{n+1} x^{n+1} \sin. x - \int \frac{1}{n+1} x^{n+1} dx \cos. x$$

$$\int x^n dx \cos. x = \frac{1}{n+1} x^{n+1} \cos. x + \int \frac{1}{n+1} x^{n+1} dx \sin. x$$

semper devenitur ad formulam $\int \frac{dx \cos. x}{x}$. Quare haec formula in analysi satis erit observabilis.

Problema IV.

Posito quod n sit quantitas negativa $= -r$; sit autem $r < 2$, & quod integrale $\int x^n dx \sin. x$ annihiletur quando $x = 0$; invenire valorem integralis pro valore x satis magno, atque etiam pro $x = \infty$.

Solutio.

Per conditionem Problematis habebitur

$$[7] \int x^n dx \sin. x = \frac{1}{n+2} x^{n+2} - \frac{1}{n+4} \cdot \frac{x^{n+4}}{2.3} + \frac{1}{n+6} \cdot \frac{x^{n+6}}{2.3.4.5} - \ldots$$

sine additione constantis. Ex aequationibus vero Problematis 1. habetur

$$[8] \int x^n dx \sin. x = -x^n \cos. x + nx^{n-1} \sin. x + n(n-1) x^{n-2} \cos. x$$
$$- n(n-1)(n-2) x^{n-3} \sin. x - \ldots\ldots$$
$$\pm n(n-1)(n-2)\ldots\ldots(n-(\mu-2)) x^{n-(\mu-1)} \tfrac{\sin.}{\cos.} x$$
$$\mp n(n-1)(n-2)\ldots\ldots(n-(\mu-1)) \int x^{n-\mu} dx \tfrac{\sin.}{\cos.} x$$

ubi μ est index terminorum ante formulam summatoriam, & ubi adhibendum est signum superius si μ fuerit formae $4p+2$, vel $4p+3$ existente p numero integro; signum vero inferius si μ fuerit formae $4p$, vel $4p+1$. Scribendum vero erit $\sin. x$ si μ fuerit par; & contra $\cos. x$ si μ fuerit impar.

Si in terminis ante formulam summatoriam loco $\cos. x$ & $\sin. x$ substituantur series, quae exhibent valorem $\cos. x$, & $\sin. x$; habebitur congeries serierum infinitarum, in quibus ob potentiam n ipsius x vel fractam vel $= -1$, quae afficit singulos terminos, nullus apparebit terminus constans, sed omnes afficientur potentiis ipsius x. Eodem modo si in formula summatoria substituantur series loco $\sin. x$, aut $\cos. x$, & integrentur singuli termini sine additione constantis; habebitur series infinita, in qua nullus erit terminus constans. Modo si aequatio [8] ita immutata comparetur cum aequatione [7]; termini affecti potentiis iisdem x iidem esse debebunt in utraque, deletis terminis, qui in [8] immutata se mutuo destruent. Ergo cum [7] in nihilum abeat quando $x = 0$; in nihilum abibit etiam [8] immutata, in qua nempe loco $\sin. x$, & $\cos. x$ substitutae sunt series, & peracta integratio formulae summatoriae sine additione constantis.

Si sumatur $x = \mu = \infty$; facile apparet terminos omnes positos ante formulam summatoriam in aequatione [8] fieri infini-

infinitesimos, & quidem successive ordinum superiorum usque ad ultimos, in quibus convergentia deficit ob factorem termini sequentis $= -\frac{\mu}{x} = -1$. Formula ergo summatoria sola evoluta sine additione constantis dabit valorem integralis $\int x^n dx \sin. x$, quem induit in casu $x = \infty$; posito quod in ipsa formula summatoria sumatur $\mu = x = \infty$.

Posito ergo quod $\int x^n dx \sin. x$ annihiletur quando $x = 0$; in casu $x = \infty = \mu$ erit

$$[9] \int x^n dx \sin. x = \mp n(n-1)(n-2)\ldots(n-(\mu-1)) \int x^{n-\mu} dx \,{}^{\sin.}_{\cos.}\, x$$

peracta integratione in secundo membro aequationis per substitutionem serierum sine additione constantis.

Si vero $x = \mu + \rho$, existente ρ fractione, sit tantum quantitas satis magna, neque tamen infinita; termini ante formulam summatoriam constituent seriem satis convergentem, quae addita valori formulae summatoriae evolutae per series dabit valorem integralis $\int x^n dx \sin. x$, qui quando x est quantitas satis magna, per aequationem [7] haberi non potest.

Sumatur μ formae $4p$; erit

$$\int x^n dx \sin. x = + n(n-1)(n-2)\ldots\ldots(n-(\mu-1))$$

$$\int x^{n-\mu} dx \sin. x = + n(n-1)(n-2)\ldots(n-(\mu-1))$$

$$\int x^{n-\mu} dx \left[x - \frac{x^3}{2.3} + \frac{x^5}{2.3.4.5} - \ldots\right] = + n(n-1)(n-2)\ldots(n-(\mu-1))$$

$$\left[\frac{x^{n-\mu+2}}{n-\mu+2} - \frac{1}{2.3}.\frac{x^{n-\mu+4}}{n-\mu+4} + \frac{1}{2.3.4.5}.\frac{x^{n-\mu+6}}{n-\mu+6} - \ldots\right.$$

$$\left.\pm \frac{1}{2.3.4\ldots(2\nu-1)}.\frac{x^{n-\mu+2\nu}}{n-\mu+2\nu} \mp \ldots\right],$$ ubi ν est index terminorum, qui quando est impar, adhibetur signum superius; quando vero est par, signum inferius.

 Si

Si nunc in termino generali $\pm \frac{1}{2.3.4\ldots(2\nu-1)}$. $\frac{x^{n-\mu+2\nu}}{n-\mu+\nu}$ sumatur $2\nu=\mu$, ac multiplicetur ille terminus per $n(n-1)(n-2)\ldots(n-(\mu-1))$ coefficientem seriei habebimus terminum $-\frac{n}{1}.\frac{(n-1)}{2}.\frac{(n-2)}{3}\ldots\frac{(n-(\mu-2))}{\mu-1}.\frac{(n-(\mu-1))x^n}{n}=-\frac{T}{n}$, posito quod fiat $\frac{n}{1}.\frac{(n-1)}{2}.\frac{(n-2)}{3}\ldots\frac{(n-(\mu-2))}{\mu-1}(n-(\mu-1))x^n=T$; termini vero sequentes ad dexteram erunt

$$+\frac{T}{n+2}.\frac{x^2}{\mu(\mu+1)}-\frac{T}{n+4}.\frac{x^4}{\mu(\mu+1)(\mu+2)(\mu+3)}+\ldots\ldots$$

termini vero retrocedendo ad sinistram ipsius $-\frac{T}{n}$ erunt

$$+\frac{T}{n-2}.\frac{(\mu-2)(\mu-1)}{x^2}-\frac{T}{n-4}.\frac{(\mu-4)(\mu-3)(\mu-2)(\mu-1)}{x^4}+\ldots\ldots$$

quae duae series posito quod μ sit aequalis ipsi x, aut parum distet, erunt convergentes, atque inservient simul cum serie posita ante formulam summatoriam in aequatione [8] ad habendum valorem integralis $\int x^n dx \,\text{sin.}\, x$ quando x est quantitas satis magna.

Quando x est adhuc quantitas finita; series sinistra habebit numerum finitum terminorum; series vero dextra semper abibit in infinitum.

Sit nunc $x=\mu=\infty$. Cum in hoc casu sit $\frac{x^2}{\mu(\mu+1)}=1$; $\frac{(\mu-2)(\mu-1)}{x^2}=1$; &c., series ad dextram fiet

$$\frac{T}{n+2}-\frac{T}{n+4}+\frac{T}{n+6}-\&c. \ldots\ldots$$ series vero ad sinistram

T

$\frac{T}{n-2} - \frac{T}{n-4} + \frac{T}{n-6} - \&c. \ldots$ Quare valor integralis propositi, qui in casu $x = \infty$ definitur per solam formulam summatoriam, habebitur per aequationem

$$\int x^n \,\text{sin.}\, x = T. \begin{cases} \frac{1}{2+n} - \frac{1}{4+n} + \frac{1}{6+n} - \ldots \\ \frac{1}{2-n} - \frac{1}{2-n} + \frac{1}{4-n} - \ldots \end{cases}$$

existente $T = \frac{n}{1} \cdot \frac{(n-1)}{2} \cdot \frac{(n-2)}{3} \ldots \frac{(n-(\mu-2))}{\mu-1} (n-(\mu-1))\mu^n$.

Cum autem in T coefficientes $n, (n-1), (n-2)$, &c. usque ad $(n-(\mu-1))$ inclusive sit numero pares, sunt enim μ, qui numerus sumptus est formae $4p$; erit T quantitas positiva.

Sed est $\frac{1}{2+n} - \frac{1}{4+n} + \frac{1}{6+n} - \ldots = \int \frac{u^{1+n}\, du}{1+u^2}$ posito post integrationem $u = 1$;

est etiam $\frac{1}{-n} - \frac{1}{2-n} + \frac{1}{4-n} - \ldots = \int \frac{u^{-1-n}\, du}{1+u^2}$;

erit itaque $\int x^n dx \,\text{sin.}\, x = T \int \frac{u^{1+n} + u^{-1-n}}{1+u^2} du$; posito post integrationem $u = 1$

Corollarium I.

Peculiarem attentionem meretur formula $\int \frac{dx \,\text{sin.}\, x}{x}$, quae analoga est formulae superius consideratae $\int \frac{dx \,\text{cos.}\, x}{x}$. Cum in formula $\int \frac{dx \,\text{sin.}\, x}{x}$ sit $n = -1$; erit eius valor in

casu

casu $x=\infty$ expressus per T. $\left\{\begin{matrix} 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots\ldots \\ 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots\ldots \end{matrix}\right.$

sed pro casu $n=-1$ est $T=1$; est autem

$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots=\frac{\pi}{4}$; Erit ergo in casu $x=\infty$,

$\int\frac{dx\,\text{sin.}\,x}{x}=\frac{\pi}{2}$ posito quod pro casu $x=0$ sit

$\int\frac{dx\,\text{sin.}\,x}{x}=0$. Quod etiam resultat ex aequatione

$\int x^n\,dx\,\text{sin.}\,x=T\int\frac{u^{1+n}+u^{-1-n}}{1+u^2}\,du$, quae fit

$\int\frac{dx\,\text{sin.}\,x}{x}=\int\frac{2du}{1+u^2}=\frac{\pi}{2}$

Corollarium II.

Si ergo formulae $\int\frac{dx\,\text{cos.}\,x}{x}$, $\int\frac{dx\,\text{sin.}\,x}{x}$, $\int\frac{dx\,e^{-x}}{x}$ integrentur sine additione constantium per series, quae exhibent valores ipsius cos. x, sin. x, & e^{-x}; erit in casu $x=\infty$

$$\int\frac{dx\,\text{cos.}\,x}{x}=lx-\frac{x^2}{2.2}+\frac{x^4}{2.3.4.4}-\frac{x^6}{2.3.4.5.6.6}+\ldots=-A$$

$$\int\frac{dx\,\text{sin.}\,x}{x}=x-\frac{x^3}{2.3.3}+\frac{x^5}{2.3.4.5.5}-\frac{x^7}{2.3.4.5.6.7.7}+\ldots=\frac{\pi}{2}$$

$$\int\frac{dx\,e^{-x}}{x}=lx-x+\frac{x^2}{2.2}-\frac{x^3}{2.3.3}+\frac{x^4}{2.3.4.4}-\ldots=-A$$

Corolla-

Corollarium III.

Sit $n = -\frac{1}{2}$; erit $\frac{n}{1} = -\frac{1}{2}$; $\frac{n-1}{2} = -\frac{3}{4}$; $\frac{n-2}{2} = -\frac{5}{6}$, ac proinde

$$T = \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{7}{8} \cdots \frac{2\mu-3}{2\mu-2} (\mu - \tfrac{1}{2}) \mu^{-\frac{1}{2}} =$$

$$\frac{1.3.5.7 \ldots (2\mu-1)}{2.4.6.8 \ldots \quad 2\mu} \sqrt{\mu} = \frac{1.3.5.7 \ldots (2\mu-1)}{2.4.6.8 \ldots \quad \sqrt{2\mu}} \times \frac{1}{\sqrt{2}}.$$

Itaque cum ſit per Theorema Walliſianum $\frac{\pi}{2} = \frac{2.2.4.4.6.6.8 \ldots}{3.3.5.5.7.7 \ldots}$; erit $T = \sqrt{\frac{2}{\pi}} \times \frac{1}{\sqrt{2}} = \sqrt{\frac{1}{\pi}}$.

Nunc ut integretur $\int \frac{u^{\frac{1}{2}} + u^{-\frac{1}{2}}}{1 + u^2} du$, fiat $u = z^2$; erit

$$\int \frac{u^{\frac{1}{2}} + u^{-\frac{1}{2}}}{1 + u^2} du = 2 \int \frac{1 + z^2}{1 + z^4} dz = \int \frac{dz}{1 + 2z\sqrt{\frac{1}{2}} + zz} +$$

$$\int \frac{dz}{1 - 2z\sqrt{\frac{1}{2}} + zz} = 2\sqrt{2} \times \text{Arc. tang.} \frac{z\sqrt{\frac{1}{2}}}{1 + z\sqrt{\frac{1}{2}}} +$$

$2\sqrt{2} \times \text{Arc. tang.} \frac{z\sqrt{\frac{1}{2}}}{1 - z\sqrt{\frac{1}{2}}}$. Et quoniam ſumi debet $u = 1$, ac proinde etiam $z = 1$; erit $\int \frac{u^{\frac{1}{2}} + u^{-\frac{1}{2}}}{1 + u^2} du =$

$$2\sqrt{2} \,.\, \text{Arc.}\ 22^\circ\ 30' + 2\sqrt{2} \,.\, \text{Arc.}\ 67^\circ\ 30' = \pi\sqrt{2}.$$

Erit ergo tandem $\int \frac{dx \,\text{ſin.}\, x}{\sqrt{x}} = \sqrt{2\pi}$.

Pro-

Problema V.

Posito quod n sit quantitas negativa $= -r$, sit autem $r < 1$, & quod integrale $\int x^n\, dx \cos. x$ annihiletur quando $x = 0$; invenire valorem integralis pro valore x satis magno, ac infinito.

Solutio.

Ex conditione Problematis habebitur sine additione constantis

$$[9]\quad \int x^n\, dx \cos. x = \frac{1}{n+1} x^{n+1} - \frac{1}{n+3} \cdot \frac{x^{n+3}}{2} + \frac{1}{n+5} \cdot \frac{x^{n+5}}{2.3.4} - \frac{1}{n+7} \cdot \frac{x^{n+7}}{2.3.4.5.6.7} + \&c.$$

atque item ex aequationibus Problematis I.

$$[10]\quad \int x^n\, dx \cos. x = x^n \sin. x + n x^{n-1} \cos. x - n(n-1) x^{n-2} \sin. x - n(n-1)(n-2) x^{n-3} \cos. x + \ldots\ldots\ldots \pm n(n-1)(n-2) \ldots. (n-(\mu-2)) x^{n-(\mu-1)} \substack{\sin.\\ \cos.} x \mp n(n-1)(n-2) \ldots. (n-(\mu-1)) \int x^{n-\mu}\, dx \substack{\sin.\\ \cos.} x$$

ubi μ est index terminorum ante formulam summatoriam, & ubi adhibendum est signum superius, si μ fuerit formae $4p+1$, aut $4p+2$; signum vero inferius, si μ fuerit formae $4p$, aut $4p+3$. Scribendum vero erit $\sin. x$, si μ fuerit impar; $\cos. x$, si par.

Si in aequatione [10] integretur formula summatoria secundi membri per substitutionem serierum sine additione constantis; evanescet ipsum secundum membrum in casu $x = 0$, quo casu evanescit etiam aequatio [9]. Quod eodem modo demonstratur, quo superius demonstrata fuit evanescentia simultanea aequationum [7], & [8].

Si

Si ſumatur $x=\mu=\infty$; evaneſcet in aequatione [10] ſeries ſecundi membri ante formulam ſummatoriam ac pro inde erit

$$\int x^n\, dx \cos. x = \mp n(n-1)(n-2)\ldots(n-(\mu-1))\int x^{n-\mu}\, dx \overset{\text{ſin.}}{\underset{\text{coſ.}}{}} x$$

peracta integratione per ſeries ſine additione conſtantis.

Si vero ſit x quantitas ſatis magna finita $=\mu+\rho$; erit convergens ſeries ante formulam ſummatoriam, quae addita ſeriei ortae ex integratione ipſius formulae ſine additione conſtantis dabit integrale $\int x^n\, dx \cos. x$, quod per aequationem [9] haberi non poſſet.

Sumatur μ formae $4p$; erit

$$+n(n-1)(n-2)\ldots(n-(\mu-1))\int x^{n-\mu}\, dx \cos. x =$$

$$+n(n-1)(n-2)\ldots(n-(\mu-1))\Big[\frac{x^{n-\mu+1}}{n-\mu+1}-\frac{1}{2}\cdot\frac{x^{n-\mu+3}}{n-\mu+3}$$

$$+\frac{1}{2.3.4}\cdot\frac{x^{n-\mu+5}}{n-\mu+5}-\ldots\pm\frac{1}{2.3.4\ldots(2\nu-2)}\cdot\frac{x^{n-\mu+(2\nu-1)}}{n-\mu+(2\nu-1)}\pm\ldots\Big]$$

ubi ν eſt index terminorum; qui quando eſt impar, adhibetur ſignum ſuperius; quando vero eſt par, ſignum inferius.

Si nunc ſumatur $2\nu=\mu$; terminus generalis ductus in coefficientem ſeriei dabit terminum

$$-\frac{n}{1}\cdot\frac{(n-1)}{2}\cdot\frac{(n-2)}{3}\cdots\frac{(n-(\mu-3))}{\mu-2}\cdot\frac{(n-(\mu-2))}{x}\cdot\frac{(n-(\mu-2))\,x^n}{\mu-1},$$

qui fiat $=\dfrac{-V}{n-2}$; erunt termini ſequentes ad eius dexteram

$$+\frac{V}{n+1}\cdot\frac{x^2}{(\mu-1)\mu}-\frac{V}{n+3}\cdot\frac{x^4}{(\mu-1)\mu(\mu+1)(\mu+2)}+\ldots.$$

termini vero retrocedentes ae ſiniſtram

$$+\frac{V}{n-3}\cdot\frac{(\mu-3)(\mu-2)}{x^2}-\frac{V}{n-5}\cdot\frac{(\mu-5)(\mu-4)(\mu-3)(\mu-2)}{x^4}+\ldots,$$

quae duae ſeries poſito quod μ ſit aequalis x, aut ab ea quantitate parum diſtet, erunt convergentes, atque inſervient

ſimul

ſimul cum ſerie praecedente in aequatione [10] ad habendum integrale pro valore x ſatis magno.

Sit nunc $x = \mu = \infty$; erit

$$\int x^n dx \cos. x = V. \begin{cases} + \frac{1}{1+n} - \frac{1}{3+n} + \frac{1}{5+n} - \dots \\ + \frac{1}{1-n} - \frac{1}{3-n} + \frac{1}{5-n} - \dots \end{cases}$$

eſt autem $\frac{1}{1+n} - \frac{1}{3+n} + \frac{1}{5+n} - \dots = \int \frac{u^n du}{1+u^n}$

$$\frac{1}{1-n} - \frac{1}{3-n} + \frac{1}{5-n} - \dots = \int \frac{u^{-n} du}{1+u^2}$$

poſito poſt integrationem $u = 1$. Quare erit

$$\int x^n dx \cos. x = V \int \frac{u^n + u^{-n}}{1+u^2} du$$

Scholion.

Cum ſit $V = \frac{n}{1} \cdot \frac{(n-1)}{2} \cdot \frac{(n-2)}{3} \dots \frac{(n-(\mu-3))}{\mu-2} \cdot \frac{(n-(\mu-2))}{n} (n-(\mu-1)) x^n$,

ſit vero in ſuperiore Problemate

$$T = \frac{n}{1} \cdot \frac{(n-1)}{2} \cdot \frac{(n-2)}{3} \dots \frac{(n-(\mu-2))}{\mu-1} (n-(\mu-1)) x^n;$$

ſumpto utrinque μ formae $4p$, atque x poſitivo, erit tam V, quam T quantitas poſitiva, ac ſumpto inſuper

$x = \mu = \infty$, erit $V = T =$

$\frac{-n}{1} \cdot \frac{(1-n)}{2} \cdot \frac{(2-n)}{3} \dots \frac{(\mu-n)}{\mu+1} \mu^{n+1}$; ubi μ iam poterit eſſe numerus cuiuſvis formae

Corolla-

Corollarium.

Si fit $n = -\frac{1}{2}$; erit $V = T = \sqrt{\frac{1}{\pi}}$ (Coroll. 3. Probl. IV.);

erit autem $\int \frac{u^n + u^{-n}}{1+u^2} du = \int \frac{u^{\frac{1}{2}} + u^{-\frac{1}{2}}}{1+u^2} du = \pi\sqrt{2}$.

Erit ergo quando $x = \infty$, $\int \frac{dx \operatorname{cof.} x}{\sqrt{x}} = \sqrt{2\pi} = \int \frac{dx \operatorname{fin.} x}{\sqrt{x}}$

Solutio Problematis Euleriani per ſuperiora.

Ut apparent uſus methodi nuper traditae, iuvat revocare ad formulas ſuperiores Problema propoſitum ab Auctore in *Additamento de Curvis Elaſticis*, quod ſubiunxit praeſtantiſſimo Operi *Methodi inveniendi lineas curvas maximi, minimive proprietate gaudentes*. Ibi num. 51. haec habet. „ Non exiguum Analyſis incrementum capere exiſtimanda „ erit, ſi quis methodum inveniret, cuius ope ſaltem vero „ proxime valor horum integralium $\int dx \operatorname{fin.} \frac{ss}{2aa}$, & „ $\int ds \operatorname{cof.} \frac{ss}{2aa}$ aſſignari poſſet caſu quo s ponitur infinitum, „ quod problema non indignum videtur, in quo Geometrae „ vires ſuas exerceant “.

Nunc poſito $\frac{ss}{2aa} = x$, habebitur $s = a\sqrt{2x}$; $ds = \frac{adx}{\sqrt{2x}}$;

ac proinde $\int ds \operatorname{fin.} \frac{ss}{2aa} = \frac{a}{\sqrt{2}} \int \frac{dx \operatorname{fin.} x}{\sqrt{x}}$;

$\int ds \operatorname{cof.} \frac{ss}{2aa} = \frac{a}{\sqrt{2}} \int \frac{dx \operatorname{cof.} x}{\sqrt{x}}$. Cum ergo in caſu $x = \infty$ ſupra

invenerimus $\int \frac{dx \sin . x}{\sqrt{x}} = \int \frac{dx \cos . x}{\sqrt{x}} = \sqrt{2\pi}$; erit in casu s item infiniti $\int ds \sin . \frac{ss}{2aa} = a\sqrt{\pi} = \int ds \cos . \frac{ss}{2aa}$

Observationes in valorem T *posito* $\mu = \infty$.

Cum valor log. T exhibeatur a curva, quam Eulerus appellat satis memorabilem, eius consideratio non est hoc loco omittenda.

Posito, quod sit n quantitas negativa $= -r$, sit autem $r < 1$; $\mu = \infty$, habebitur ex superioribus

$$T = \frac{r}{1} . \frac{(1+r)}{2} . \frac{(2+r)}{3} . \frac{(3+r)}{4} \ldots\ldots \frac{(\mu+r)}{\mu+1} \mu^{1-r}$$

sive facto $r = 1 - m$; erit

$$T = \frac{(1-m)}{1} . \frac{(2-m)}{2} . \frac{(3-m)}{3} . \frac{(4-m)}{4} \ldots\ldots \frac{(\mu-m)}{\mu} \mu^{m}$$

Modo si sit $m = 1$; perspicuum est fore $T = 0$

$m = \frac{1}{2}$; demonstratum est fore $T = \sqrt{\frac{1}{\pi}}$

$m = 0$; perspicuum est fore $T = 1$

Ad perspiciendos valores T pro valoribus intermediis m habebitur

$$lT = \begin{cases} l(1-m) = -m - \frac{1}{2} m^2 - \frac{1}{3} m^3 - \frac{1}{4} m^4 - \ldots\ldots \\ +l(2-m) - l2 = -\frac{m}{2} - \frac{1}{2}\left(\frac{m}{2}\right)^2 - \frac{1}{3}\left(\frac{m}{2}\right)^3 - \frac{1}{4}\left(\frac{m}{2}\right)^4 - \ldots\ldots \\ +l(3-m) - l3 = -\frac{m}{3} - \frac{1}{2}\left(\frac{m}{3}\right)^2 - \frac{1}{3}\left(\frac{m}{3}\right)^3 - \frac{1}{4}\left(\frac{m}{3}\right)^4 - \ldots\ldots \\ \ldots\ldots\ldots \\ +l(\mu-m) - l\mu = -\frac{m}{\mu} - \frac{1}{2}\left(\frac{m}{\mu}\right)^2 - \frac{1}{3}\left(\frac{m}{\mu}\right)^3 - \frac{1}{4}\left(\frac{m}{\mu}\right)^4 - \ldots\ldots \\ + ml\mu = + ml\mu \end{cases}$$

Sumptis

Sumptis columnis verticalibus, cum sit

$$ml\mu = m\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{\mu}\right) - m\,A,$$ existente

A = 0, 577215.... ut supra; habebitur

$$(L)\quad lT = \begin{cases} - m\,A \\ - \frac{m^2}{2}\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\right. \\ - \frac{m^3}{3}\left[1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \ldots\right. \\ - \frac{m^4}{4}\left[1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \ldots\right. \end{cases}$$

atque ita in infinitum.

Ex hac aequatione apparet lT fore semper negativum, ac proinde T quantitatem fractam, quae pro valoribus intermediis ipsius m habebit valores intermedios.

Nunc vero videamus quomodo lT exhibeatur per quadraturam curvae Eulerianae.

Auctor in Opusculo, cui titulus: *Dilucidationes in Capita postrema Calculi mei Differentialis de functionibus inexplicabilibus*, quod primo editum est a Cl. Speronio in sua edit. Ticinensi Calculi Differentialis Euleriani fusius evolvit curvam huius aequationis

$$y = \frac{x}{x+1} + \frac{x}{2(x+2)} + \frac{x}{3(x+3)} + \frac{x}{4(x+4)} + \text{\&c. in inf.}\quad (M),$$

ubi quotiescumque pro x accipitur numerus integer positivus, y exprimitur finite per aequationem

$$y = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots\ldots + \frac{1}{x}\quad (N)$$

si vero x sit numerus alius quicumque; functio ipsius x expressa per aequationem (N) est inexplicabilis.

Quadraturam vero huius curvae invenit per aequationem

$$(P)\ \int y\,dx =$$

$$+\frac{x^2}{2}\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots\right)=+0,822467.x^2$$

$$-\frac{x^3}{3}\left(1+\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{4^3}+\ldots\right)=-0,400685.x^3$$

$$+\frac{x^4}{4}\left(1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\ldots\right)=+0,270581.x^4$$

$$-\frac{x^5}{5}\left(1+\frac{1}{2^5}+\frac{1}{3^5}+\frac{1}{4^5}+\ldots\right)=-0,207385.x^5$$

+ &c. in infinitum.

Erit itaque $lT = -mA - \int y\,dx$ sumpto post integrationem $x = -m$.

Liceat mihi hoc loco nonnullas cyphras in Auctoris calculis ad veritatem deducere.

Quaerit Auctor in aequatione (P) valorem integralis posito $x = 1$; ac per quaedam artificia illud reperit $= 0,577190$. Est autem revera $= 0,577215\ldots$ sive $= A$ numero iam saepissime considerato.

Nam resumatur quadratio huius curvae. Cum ergo sit

$$y = \begin{matrix} 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\&c. \\ -\frac{1}{x+1}-\frac{1}{x+2}-\frac{1}{x+3}-\frac{1}{x+4}-\frac{1}{x+5}-\&c. \end{matrix} \qquad (K)$$

in qua aequatione annihilantur simul x, & y; erit

$$\int y\,dx = \text{Const.} + x + \frac{1}{2}x + \frac{1}{3}x + \frac{1}{4}x + \frac{1}{5}x + \&c.$$
$$- l(x+1) - l(x+2) - l(x+3) - l(x+4) - l(x+5) - \&c.$$

ubi cum ita Constans determinari debeat, ut casu $x = 0$ area evanescat, integrale ita rite exprimetur

$$(Q)\ \int y\,dx =$$

$$x+\frac{x}{2}+\frac{x}{3}+\frac{x}{4}+\frac{x}{5}+\&c.\ldots \qquad -l$$

$$-l(1+x)-l\left(1+\frac{x}{2}\right)-l\left(1+\frac{x}{3}\right)-l\left(1+\frac{x}{4}\right)-l\left(1+\frac{x}{5}\right)-\&c...;$$

ac posito $x=1$ erit

$$\int ydx\; 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\ldots\ldots+\frac{1}{n}$$
$$-l2-l\frac{3}{2}-l\frac{4}{3}-l\frac{5}{4}-l\frac{6}{5}-\ldots-l\frac{n}{n-1}$$

ubi n est numerus infinitus; itaque erit

$\int ydx=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\ldots+\frac{1}{n}-ln$. Est autem

$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\ldots+\frac{1}{n}=ln+A$. Erit ergo

posito $x=1$ valor integralis $\int ydx=A$.

Idem resultat ex consideratione alterius functionis inexplicabilis $S=1.2.3.4\ldots\ldots x$, quam Auctor examinat in exemplo secundo §. 384. Calculi Different. Part. II. Cap. XVI., pro qua invenit aequationem

$$lS=-x.0,5772156649015325$$
$$+\frac{1}{2}xx\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\ldots\right)$$
$$-\frac{1}{3}x^3\left(1+\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{4^3}+\frac{1}{5^3}+\ldots\right)$$
$$+\frac{1}{4}x^4\left(1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\frac{1}{5^4}+\ldots\right)$$
$$\&c.$$

Posito enim $x=1$; quo fit etiam $S=1$; erit $lS=0$, ac proinde $0,5772156649015325$ aequale integrali aequationis (P) in eadem suppositione $x=1$.

Inde etiam sequitur, quod sumpto $x=-m$ erit $-lS=lT$;

ac proinde $S=\frac{1}{T}$, seu

$$1.2.3.4\ldots\ldots x=\frac{1}{1+x}\cdot\frac{2}{2+x}\cdot\frac{3}{3+x}\cdot\ldots\ldots\frac{\mu}{\mu+x}\mu^x.$$

Ad-

Adnotatio III. ad Cap. VIII. Sect. I. Vol. I.

Ad calcem huius capitis §. 355. haec habet Auctor de formula $\int \frac{z^{m-1}dz}{1+z^n}$ „ Caeterum omni attentione dignum „ quod hic ostendimus, formulae integralis $\int \frac{z^{m-1}dz}{1+z^n}$ valo- „ rem casu $z=\infty$ tam concinne exprimi ut sit $\frac{\pi}{n \sin. \frac{m}{n}\pi}$, „ cuius demonstratio cum per tot ambages sit adstructa me- „ rito suspicionem excitat eam via multa faciliori confici „ posse, etiamsi modus nondum perspiciatur. Id quidem ma- „ nifestum est hanc demonstrationem ex ratione sinuum an- „ gulorum multiplorum peti oportere; & quoniam in intro- „ ductione sin. $\frac{m}{n}\pi$ per productum infinitorum factorum ex- „ pressi, mox videbimus inde eandem veritatem multo faci- „ lius deduci posse, etiamsi ne hanc quidem viam pro ma- „ xime naturali haberi velim“.

Via maxime naturalis haec videri potest, quae est brevissima. In Introductione in Analysin Infinit. Lib. I. Cap. X. §. 181. invenit Auctor esse

$$\frac{\pi}{n \sin. \frac{m\pi}{n}} = \frac{1}{m} + \frac{2m}{nn-mm} - \frac{2m}{4nn-mm} + \frac{2m}{9nn-mm} - \frac{2m}{16nn-mm}$$

$$+ \frac{2m}{25nn-mm} - \ldots\ldots$$

Est

Eſt autem $\int \frac{z^{m-1}dz}{1+z^n} = \int \frac{z^{m-n-1}}{1+z^{-n}} =$

$$\int z^{m-n-1}dz\left(1 - z^{-n} + z^{-2n} - z^{3n} + \&c..\right) =$$

$$(1)\ C + \frac{1}{m-n}z^{m-n} - \frac{1}{m-2n}z^{m-2n} + \frac{1}{m-3n}z^{m-3n} - \ldots,$$

ubi C ita ſumenda eſt ut poſito $z=0$ ſeries annihiletur ex conditione, quam Eulerus poſuit (§. 351.).

Eſt rurſus

$$\int \frac{z^{m-1}dz}{1+z^n} = \int z^{m-1}dz\left(1 - z^n + z^{2n} - z^{3n} + z^{4n} - \ldots\right) =$$

$$(2)\ \frac{1}{m}z^m - \frac{1}{m+n}z^{m+n} + \frac{1}{m+2n}z^{m+2n} - \frac{1}{m+3n}z^{m+3n} + \ldots$$

in qua aequatione nulla conſtans additur, cum pro valoribus poſitivis m, & n (§. 351. & 77.) poſito $z=0$ ſeries ipſa annihiletur.

Ob (1) = (2) habebitur ſumpto $z=1$

$$C - \frac{1}{n-m} + \frac{1}{2n-m} - \frac{1}{3n-m} + \frac{1}{4n-m} - \ldots\ldots$$

$$= \frac{1}{m} - \frac{1}{n+m} + \frac{1}{2n+m} - \frac{1}{3n+m} + \ldots\ldots$$

ac proinde $C = \frac{\pi}{n \sin. \frac{m\pi}{n}}$

Cum vero ex conditione Euleri (§. 351. & 77.) ſit $m < n$; ſi in aequatione (1) ſumatur $z = \infty$ habebitur

$$\int \frac{z^{m-1}dz}{1+z^n} = C = \frac{\pi}{n \sin. \frac{m\pi}{n}}$$

quod erat demonſtrandum.

Scholion.

Scholion.

Eadem facilitate definitur casu $z = \infty$ esse

$$\int \frac{z^{m-1}dz}{1-z^n} = \frac{\pi}{n \text{ tang. } \frac{m}{n}\pi} \text{; est enim}$$

$$\int \frac{z^{m-1}dz}{1-z^n} = -\int \frac{z^{m-n-1}dz}{1-z^{-n}} =$$

$$-\int z^{m-n-1}dz\left(1+z^{-n}+z^{-2n}+z^{-3n}+\ldots\ldots\right) =$$

$$(3)\ K - \frac{1}{m-n}z^{m-n} - \frac{1}{m-2n}z^{m-2n} - \frac{1}{m-3n}z^{m-3n} - \ldots$$

ubi K ita sumenda est, ut integrale evanescat posito $z = 0$.

Est rursus $\int \frac{z^{m-1}dz}{1-z^n} = \int z^{m-1}dz\left(1+z^n+z^{2n}+z^{3n}+\ldots\right) =$

$$(4)\ \frac{1}{m}z^m + \frac{1}{m+n}z^{m+n} + \frac{1}{m+2n}z^{m+2n} + \frac{1}{m+3n}z^{m+3n} + \ldots,$$

in qua aequatione nulla constans addenda est.

Ob (3) = (4) habebitur sumpto $z = 1$

$$K - \frac{1}{m-n} - \frac{1}{m-2n} - \frac{1}{m-3n} - \ldots$$

$$= \frac{1}{m} + \frac{1}{m+n} + \frac{1}{m+2n} + \frac{1}{m+3n} + \ldots\ldots$$

$$K = \frac{1}{m} - \frac{2m}{nn-mm} - \frac{2m}{4nn-mm} - \frac{2m}{9nn-mm} - \ldots$$

seu $K = \frac{\pi}{n \text{ tang. } \frac{m}{n}\pi}$ (Introd. in Analys. Inf. Lib. I. Cap. X. §. 181.). Quare ex aequatione (3) posito $z = \infty$ habebitur

bebitur $\int \frac{z^{m-1}\,dz}{1-z^n} = K = \frac{\pi}{n \text{ tang.} \frac{m}{n}\pi}$, quod Theorema consociatur Theorematì Euleriano.

Adnotatio IV. ad Cap. I. Sect. II. Vol. I.

Ad §. 431. propositam aequationem differentialem

$$\alpha y dx + \beta x dy + x^m y^n (\gamma y dx + \delta x dy) = 0$$

dividendo per xy, ac adhibitis substitutionibus $x^\alpha y^\beta = t$; $x^\gamma y^\delta = u$ transformat in aequationem

$$t^{\frac{\gamma n - \delta m}{\alpha\delta - \beta\gamma} - 1} dt + u^{\frac{\alpha n - \beta m}{\alpha\delta - \beta\gamma} - 1} du = 0;$$

cuius aequationis integrale est

$$\frac{t^{\frac{\gamma n - \beta m}{\alpha\delta - \beta\gamma}}}{\gamma n - \delta m} + \frac{u^{\frac{\alpha n - \beta m}{\alpha - \beta m}}}{\alpha n - \beta m} = C$$

ubi tantum superest ut restituantur valores t, & u. Deinde notat „ si fuerit vel $\gamma n - \delta m = 0$, vel $\alpha n - \beta m = 0$, loco illorum membrorum vel lt, vel lu scribi debere ".

Notandus vero est etiam alius casus, in quo aequatio integralis non exhibet valorem, qui satisfaciat aequationi differentiali propositae; qui tunc accidit cum habetur $\alpha\delta - \beta\gamma = 0$, quo in casu variabiles t, & u haberent infinitum pro exponente. Sed in eo casu posito $\alpha = c\gamma$ est $\beta = c\delta$; ac proinde aequatio differentialis proposita abit in sequentem

$$(x^m y^n + c)\ (\gamma y dx + \delta x dy) = 0$$

cui aequationi satisfacit aequatio $x^m y^n + c = 0$; tum alia $\gamma y dx + \delta x dy = 0$, cuius integrale est $y x^{\frac{\gamma}{\delta}} = \text{Const.}$

Ad §. 433. propositam aequationem differentialem

$$y dy + dy(a + bx + nxx) = y dx(c + nx)$$

 per

per ſubſtitutionem $u = \frac{dy}{dx}$ reducit ad ſeparationem variabilium, perveniendo ad aequationem differentialem

$$\frac{dx}{(a + bx + nxx)(c + nx)} = \frac{du}{u(na + cc - bc + (b - 2c)u + uu)}$$

cuius integratio per logarithmos, & angulos abſolvi poteſt. Subdit vero: „ Caſu autem hic vix praevidendo evenit ut „ haec ſubſtitutio ad votum ſucceſſerit, neque hoc problema „ magnopere iuvabit “.

Non apparet, cur Eulerus ſuum problema contempſerit. Nam

I.° in eius aequatione continetur aequatio

$$2ANdM - AMdN = M(MdN - NdM) - 2NdN,$$

quam adhibet pro conditione integrabilitatis Cap. III. §. 498. quam cum ad hoc problema non retuliſſet, ait „ quae cum „ in nulla iam tractatarum contineatur videndum eſt quo- „ modo tractabilior reddi queat “. Sane ſi fiat

$N = y$; $M = x$; $A = -2b = -c$; $n = -\frac{1}{2}$; habebitur aequatio $ydy + dy(bx + nxx) = y\,dx(c + nx)$, quae eſt ipſa problematis facto $a = 0$.

II.° Ipſa aequatio $dy(y + A + BV + CVV) - CyVdV = 0$, quam §. 494. invenit integrabilem per multiplicatorem

$$\frac{1}{y^3 + (2A + BV)yy + A(A + BV + CVV)y}$$

continetur in aequatione §. 433. ſi fiat $A = a$; $B = b$; $C = n$; $c = 0$; $V = x$.

Adnotatio V. ad Sectionem III. Vol. I.

In hac Sectione, cui titulus: *De reſolutione aequationum differentialium, in quibus differentialia ad plures dimen-*

mensiones assurgunt, vel adeo transcendenter implicantur, quatuor problemata ponit Auctor; quibus concludit Sectionem tertiam, ac Volumen primum his verbis: „ atque hucusque fere Geometris in resolutione aequationum differentialium primi gradus etiamnum pertingere licuit ".

Cum vero Celeber. Petrus Paoli olim ante me in Ticinensi nunc in Pisano Archigymnasio Math. Prof. occasione cuiusdam elegantis Problematis Optici (in Opusc. Analyt. Liburni 1780. Opusc. IV.) incidisset in aequationem

$$\frac{x\,dy - y\,dx}{\surd(x^2+y^2)\,\surd(dx^2+dy^2)} = 1;$$

quam videbat per nullas ex methodis Eulerianis integrari posse; excogitavit substitutionem quamdam, per quam non solum proposita formula, sed infinitae aliae reduci possunt ad separationem variabilium, atque adeo ad integrationem, ex qua nos adhuc magis generales formulas eliciemus.

Quantum vero methodus substitutionum, atque separationis variabilium excoli debeat in resolutione huiusmodi aequationum differentialium, in quibus differentialia ad plures dimensiones assurgunt, vel inde intelligi potest quod Eulerus, cui adeo opportuna videtur methodus multiplicatoris pro integratione aequationum differentialium supra methodum separationis variabilium; de his aequationibus haec habet §. *677*. „ Altera vero methodus, qua supra usi sumus, quaerendo „ factorem, qui aequationem differentialem reddat per se integrabilem hic plane locum non habet, cum per differentiationem aequationis finitae numquam differentialia ad plures dimensiones exsurgere queant ". Proponatur ergo in genere.

Problema.

Invenire casus, in quibus aequatio $\frac{x\,dy - y\,dx}{\surd(dx^2+dy^2)} = X$, existente X functione x, & y reduci potest ad separationem variabilium.

 Solutio.

Solutio.

Fiat cum Cl. Paoli „ $x = uz$, $y = u\surd(1 - z^2)$. Iam „ si P est functio variabilium u, & z, in quam per praecedentes substitutiones vertitur X; aequatio proposita ita „ erit comparata $\frac{-u^2 dz}{\surd(u^2 dz^2 + (1 - z^2) du^2)} = P$. Fiat

$dz = pdu$, eritque $\frac{-u^2 p}{\surd(u^2 p^2 + 1 - z^2)} = P$, sive

„ $p = \frac{dz}{du} = \frac{P\surd(1 - z^2)}{u\surd(u^2 - P^2)}$ “. Fiat nunc $P = uQ$, ita ut habeatur $\frac{dz}{du} = \frac{Q\surd(1 - z^2)}{u\surd(1 - Q^2)}$, ac fiat rursus $\frac{Q}{\surd(1 - Q^2)} = VZ$, ubi V est functio ipsius u, & Z est functio ipsius z, ita ut habeatur $Q = \frac{Vz}{\surd(1 + V^2 z^2)}$; habebitur $\frac{dz}{Z\surd(1 - z^2)} = \frac{duV}{u}$, ubi variabiles sunt separatae.

Cum ergo per substitutionem $x = uz$; $y = u\surd(1 - z^2)$, habeamus $u^2 = x^2 + y^2$; $z^2 = \frac{x^2}{x^2 + y^2}$, ac possint separari variabiles cum $P = \frac{uVZ}{\surd(1 + V^2 Z^2)}$; eae poterunt separari in aequatione proposita cum erit

$$(1) \ldots \text{X} = \frac{\surd(x^2 + y^2)\text{F}, \surd(x^2 + y^2) f, \frac{x}{\surd(x^2 + y^2)}}{\surd\left(1 + \text{F}^2, \surd(x^2 + y^2) f^2, \frac{x}{\surd(x^2 + y^2)}\right)}$$

vel per conversionem variabilium facto $y = uz$; $x = u\surd(1 - z^2)$, cum erit

$\surd(x^2$

$$(2)\ldots\ldots X=\frac{\sqrt{(x^2+y^2)}F,\sqrt{(x^2+y^2)}\,f,\frac{y}{\sqrt{(x^2+y^2)}}}{\sqrt{\left(1+F^2,\sqrt{(x^2+y^2)}f^2,\frac{y}{\sqrt{(x^2+y^2)}}\right)}}$$

Si in aequatione (1) sumatur $f,\frac{x}{\sqrt{(x^2+y^2)}}=1$, habebitur casus primus Cl. Paoli; si vero in eadem aequatione sumatur F, $\sqrt{(x^2+y^2)}=1$, habebitur casus secundus.

Notandum vero est pro his aequationibus in genere, quod Cl. Paoli notavit pro sua

$\frac{xdy-ydx}{\sqrt{(x^2+y^2)^3}\sqrt{(dx^2+dy^2)}}=1$, qua continetur problema opticum ab eo assumptum curvae aequalis intensitatis luminis reflexi. Animadvertit ille quod praeter aequationem integralem $(x^2+y^2)^2=2kxy+(y^2-x^2)\sqrt{(1-k^2)}$, quae resultat ex integratione formulae $\frac{dz}{\sqrt{(1-z^2)}}=\frac{udu}{\sqrt{(1-u^4)}}$ restituendo x, & y; satisfacit problemati etiam aequatio circuli $x^2+y^2=a^2$, seu pro casu peculiari $x^2+y^2=1$, quae non videbatur erui posse ex differentiali proposita, neque continetur in integrali completo. Quod cum primum ille collegisset ex conditionibus geometricis problematis optici; postea docuit obtineri etiam ex differentiali proposita transformata per substitutiones, si nullus ex eius factoribus negligatur. Nam cum aequatio proposita per substitutionem $x=uz$; $y=u\sqrt{(1-z^2)}$ primum transformata fuerit in

$$-u^2dz=P\sqrt{(u^2dz^2+(1-z^2)du^2)}\ldots(3)$$

deinde per aliam substitutionem $pdu=dz$ in aliam $-u^2pdu=Pdu\sqrt{(u^2p^2+1-z^2)}\ldots(4)$; si non negligatur factor du, per quem tota aequatio potest dividi; sed pro una ex radicibus huius ultimae aequationis fiat $du=0$; habebitur $u=a$; $u^2=a^2=x^2+y^2$, quod valet pro casibus omnibus

omnibus in quibus aequatio proposita reduci potest ad separationem variabilium per substitutionem Cl. Paoli. Itaque si $R = 0$ sit integrale completum aequationis

$$\frac{dz}{Z\sqrt{(1-z^2)}} = \frac{V\,du}{u}$$; resolutio magis completa problematis, pro quo habetur aequatio differentialis proposita separabilis per easdem substitutiones, habebitur per aequationem

$$(x^2 + y^2 - a^2)R = 0$$

Neque tamen adhuc pro omnino completa habenda erit. Nam cum aequatio (3) per substitutionem $du = q\,dz$ transformetur in sequentem

$$-u^2\,dz = P\,dz\sqrt{(u^2 + (1-z^2)q^2)} \ldots (5);$$

hac divisa per dz adhuc obtinetur

$$q = \frac{u\sqrt{(u^2 - P^2)}}{P\sqrt{(1-z^2)}} = \frac{du}{dz}$$

prorsus ut supra; quare in idem integrale completum $R = 0$ devenimus. Sed cum aequationis (5) radix sit $dz = 0$; inde habebitur $z = b$; $z^2 = b^2 = \frac{x^2}{x^2 + y^2}$; unde habetur $y = \pm x\sqrt{\left(\frac{1-b^2}{b^2}\right)}$, quae est aequatio pro linea recta quoties b est fractio. Itaque magis adhuc completa fiet solutio problematis per aequationem

$$(x^2(b^2 - 1) + b^2y^2)(x^2 + y^2 - a^2)R = 0$$

Quod etiam aequatio rectae $y = \pm x\sqrt{\left(\frac{1-b^2}{b^2}\right)}$ exhibeat novam solutionem problematis optici curvae, ex cuius punctis omnibus lux aeque intensa reflectatur posito quod sit lucis intensitas directe ut sinus anguli incidentiae, atque inverse ut quadratum distantiae a puncto radiante; inde patet, quod facto $y = 0$, quando x est aequalis distantiae, in qua sit lucis intensitas $= 1$ habetur $b = 1$, ac proinde semper $y = 0$; unde sequitur lineam rectam transire per punctum radians.

radians. In hoc autem casu nulla est pro omnibus punctis lucis reflexae intensitas, quo ipso habito solvitur problema. Sed etiam si sit $b < 1$ cum y, & x simul annihilentur adhuc recta transibit per punctum radians, ut satisfiat conditioni problematis.

Si fiat conversio variabilium, adhuc habebimus praeter integrale aequationis $\frac{dz}{Z\sqrt{(1-z^2)}} = \frac{du\,V}{u}$ duas aequationes

$x^2 + y^2 = m^2$; $\frac{y^2}{x^2+y^2} = n^2$, pro circulo & recta. Adeo ut in genere praeter integrale aequationis transformatae ubi variabiles sunt z, & u; habeantur etiam novae solutiones per aequationes $z = a$; $u = b$.

Itaque regula habetur etiam pro aliis aequationibus differentialibus, quae ad integrationem deducuntur per transformationem variabilium. Sit in genere aequatio differentialis $P\,dx + Q\,dy = 0$, ubi P, & Q sunt functiones x, & y. Sint autem z & u tales functiones ipsarum x, & y, ut per earum substitutionem aequatio $P\,dx + Q\,dy = 0$ transformetur in aequationem $R\,dz + S\,du = 0$, ubi R, & S iam sunt functiones ipsarum z, & u; aequatio vero $R\,dz + S\,du = 0$ sit integrabilis. Praeter solutionem, quae oritur ex integrali huius aequationis rursus transformato in functionem x, & y; aequatio $P\,dx + Q\,dy = 0$ sortietur alias binas solutiones ex aequationibus $z = a$; $u = b$; per eas enim habetur $dz = 0$, $du = 0$, ac proinde $R\,dz + S\,du = 0$. Quod si forte aequatio $P\,dx + Q\,dy = 0$ etiam per substitutionem novarum variabilium p, & q ad integrationem adduci posset ope aequationis transformatae $M\,dp + N\,dq = 0$; ubi M, & N sunt functiones ipsarum p, & q; adhuc duae novae solutiones problematis haberentur per aequationes $p = c$; $q = f$; existentibus constantibus c, & f.

Cum $\frac{x\,dy - y\,dx}{\sqrt{(dx^2 + dy^2)}}$ sit perpendiculum demissum ab initio

tio abſciſſarum in tangentem curvae; ſit vero $\sqrt{(x^2+y^2)}$ radius vector; ac ſit $\frac{x}{\sqrt{(x^2+y^2)}}$ coſinus anguli radii vectoris, atque axis abſciſſarum; $\frac{y}{\sqrt{(x^2+y^2)}}$ ſinus eiuſdem anguli; apparebit ex aequationibus (1), & (2) quibus conditionibus relationis inter perpendiculum, radium vectorem eiuſque functiones, ac functiones ſinus aut coſinus anguli anomaliae haberi poſſit per ſubſtitutionem Paoli aequatio curvae.

FINIS.

			Errata.	*Corrige.*
Pag. 5.	*lin.*	7.	*adde in fine lineae*	(3)
	lin.	8.	*adde in fine lineae*	(4)
11.	*lin.*	15.	532	523
13.	*lin.*	2.	$\frac{(n-2)(n-3)}{3x^3}$	$\frac{(n-2)(n-1)n}{3x^3}$
	lin.	7.	$\frac{(n-2)(n-1)}{3(lz)^3}$	$\frac{(n-2)(n-1)n}{3(lz)^3}$
	lin.	18.	ſumantur	ſummantur
17.	*lin.*	*antepen.*	conſeri	cenſeri
32.	*lin.*	5. *aſcend.*	$\frac{3}{2.4.6}\cdot\frac{3}{6}$	$\frac{1}{2.4}\cdot\frac{3}{6}$
			$\frac{3.5}{2.4.6.8}$	$\frac{3}{2.4.6}$
57.	*lin.*	6.	apparent	appareat

ADNOTATIONUM
AD
CALCULUM INTEGRALEM
EULERI

IN QUIBUS NONNULLAE FORMULAE AB EULERO PROPOSITAE PLENIUS EVOLVUNTUR

PARS ALTERA

AUCTORE

LAURENTIO MASCHERONIO

IN R. ARCHIGYMNASIO TICINENSI MATHEM PROF.

ACAD. PATAVINAE, R. MANTUANAE

ATQUE ITALICAE SOCIO.

TICINI MDCCXCII.

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LAURENTIUS MASCHERONIUS

D. D. L. M.

ADNOTATIONES
AD
CALCULUM INTEGRALEM
EULERI.

Adnotatio altera ad Cap. IV. Sect. I. Vol. I.

Evolutio completa formulae integralis

$$\int \frac{(x^\alpha \pm x^\beta)\,dx}{lx}$$

Revocemus ad methodum ſuperius traditam in Adnot. I. ad hoc Caput, formulam Euleri T. XX. Nov. Comment. Petrop. pag. 59. Ibi Auctor haec habet: „ Cum „ nuper inveniſſem integrale huius formulae differentialis „ $\frac{(x^\alpha - x^\beta)\,dx}{lx}$, ſi ita capiatur ut evaneſcat poſito $x = 0$, „ tum vero ſtatuatur $x = 1$, aequari huic valori $l\frac{\alpha+1}{\beta+1}$: „ haec integratio eo magis attentione digna mihi videbatur, „ quod eius veritas per nullas methodos hactenus uſitatas „ oſtendi poſſet. Quamobrem nullum plane eſt dubium, „ quin ea plurimum in receſſu habeat, & ad multa alia „ praeclara inventa in Analyſi perducere queat “.

 Hacte-

Hactenus ille:

Sed non solum facile veritas huius integrationis Eulerianae per methodum superius traditam ostendi potest pro simplici casu $x = 1$; verum etiam habentur series ad exhibendum valorem integralis pro quocumque alio valore ipsius x.

Nam facto $x^{\alpha+1} = z$; habetur

$$(A)\ldots\int \frac{x^\alpha dx}{lx} = \int \frac{dz}{lz} =$$

$$A + l \mp lz + lz + \frac{(lz)^2}{2.2} + \frac{(lz)^3}{2.3.3} + \frac{(lz)^4}{2.3.4.4} + \ldots$$

ubi est $A = 0{,}577215\ 664901\ 532860\ 618112\ldots$ in termino vero $l \mp lz$ signum $-$ adhibendum est pro valoribus z minoribus unitate; signum vero $+$ pro valoribus eiusdem z unitate maioribus (Vide Adnot. I. pag. 11. & 17.).

Facto item $x^{\beta+1} = y$; habetur

$$(B)\ldots\int \frac{x^\beta dx}{lx} = \int \frac{dy}{ly} =$$

$$A + l \mp ly + ly + \frac{(ly)^2}{2.2} + \frac{(ly)^3}{2.3.3} + \frac{(ly)^4}{2.3.4.4}$$

Quare erit

$$\int \frac{(x^\alpha - x^\beta)dx}{lx} = l\frac{\mp lz}{\mp ly} + l\frac{z}{y} + \frac{(lz)^2 - (ly)^2}{2.2} + \frac{(lz)^3 - (ly)^3}{2.3.3} + \ldots$$

sive substitutis valoribus z, & y; erit

$$(a)\ldots\int \frac{(x^\alpha - x^\beta)dx}{lx} = l\frac{\alpha+1}{\beta+1} + (\alpha-\beta)lx + \frac{(\alpha+1)^2 - (\beta+1)^2}{2.2}(lx)^2$$
$$+ \frac{(\alpha+1)^3 - (\beta+1)^3}{2.3.3}(lx)^3 + \frac{(\alpha+1)^4 - (\beta+1)^4}{2.3.4.4}(lx)^4 + \ldots$$

ubi lex seriei satis est manifesta.

Ex hac aequatione statim patet casu $x = 1$ fore

$$\int \frac{(x^\alpha - x^\beta)dx}{lx} = l\frac{\alpha+1}{\beta+1}$$

Si

Si in aequatione (a) ponamus cum Eulero $\alpha = n\sqrt{-1}$; & $\beta = -n\sqrt{-1}$; habebimus $x^{\alpha} - x^{\beta} = x^{n\sqrt{-1}} - x^{-n\sqrt{-1}} = e^{nlx\sqrt{-1}} - e^{-nlx\sqrt{-1}} = 2\sqrt{-1}$ fin. nlx; quo valore fubftituto (ob $l\frac{1+n\sqrt{-1}}{1-n\sqrt{-1}} = 2\sqrt{-1}$ A. tang. n) prodit generatim

$$(b)\ldots \int \frac{dx \text{ fin. } nlx}{lx} = \text{A. tang. } n + nlx + \frac{2n(lx)^2}{2.2} + \frac{(3n-n^3)(lx)^3}{2.3.3} + \frac{(4n-4n^3)(lx)^4}{2.3.4.4} + \ldots$$

cuius feriei terminus generalis eft

$$\frac{\left(pn - \frac{p(p-1)(p-2)}{2.3}n^3 + \frac{p(p-1)(p-2)(p-3)(p-4)}{2.3.4.5}n^5 - \ldots\right)(lx)^p}{2.3.4\ldots p.p}$$

Si vero fit $x = 1$; erit

$$\int \frac{dx \text{ fin. } nlx}{lx} = \text{A. tang. } n$$

Si in aequatione (b) fumatur $n = 1$; habebitur

$$(c)\ldots \int \frac{dx \text{ fin. } lx}{lx} = \frac{\pi}{4} + lx + \frac{2(lx)^2}{2.2} + \frac{2(lx)^3}{2.3.3} + \frac{4(lx)^5}{2.3.4.5.5} - \frac{8(lx)^6}{2.3.4.5.6.6} - \frac{8(lx)^7}{2.3\ldots7.7} + \frac{16(lx)^9}{2.3\ldots9.9} + \frac{32(lx)^{10}}{2.3\ldots10.10} - \ldots$$

ubi defunt omnes potentiae divifibiles per 4.

Si poft integrationem fiat $x = 1$; erit

$$\int \frac{dx \text{ fin. } lx}{lx} = \frac{\pi}{4}.$$

Series fuperiores (a), (b), & (c) inferviunt ad habendum proxime valorem integralis quando termini convergunt; fi vero divergant; facto ut fupra $x^{\alpha+1} = z$; $x^{\beta+1} = y$;

ac

ac proinde

$$\int \frac{(x^\alpha - x^\beta)\,dx}{lx} = \int \frac{dz}{lz} - \int \frac{dy}{ly}$$

habebuntur aliae series convergentes substituendae ex aequatione (10) Adnotationis I. ad hoc caput pag. 10; erit nempe

$$(d)\ldots \int \frac{(x^\alpha - x^\beta)\,dx}{lx} =$$

$$z\left(\frac{1}{lz}+\frac{1}{(lz)^2}+2\frac{1}{(lz)^3}+2.3\frac{1}{(lz)^4}+\ldots+2.3.4\ldots(m-1)\frac{1}{(lz)^m}\right)$$

$$+A-1-\frac{1}{2}-\frac{1}{3}-\frac{1}{4}-\ldots\ldots-\frac{1}{m}$$

$$+\,l \pm lz$$

$$+\frac{lz}{m+1}+\frac{(lz)^2}{2(m+1)(m+2)}+\frac{(lz)^3}{3(m+1)(m+2)(m+3)}+\ldots\ldots\&c.$$

$$-\frac{m}{lz}-\frac{(m-1)m}{2(lz)^2}-\frac{(m-2)(m-1)m}{3(lz)^3}-\ldots\ldots\&c.$$

$$-y\left(\frac{1}{ly}+\frac{1}{(ly)^2}+2\frac{1}{(ly)^3}+2.3\frac{1}{(ly)^4}+\ldots+2.3.4\ldots(\mu-1)\frac{1}{(ly)^\mu}\right)$$

$$-A+1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots\ldots+\frac{1}{\mu}$$

$$-\,l \pm ly$$

$$-\frac{ly}{\mu+1}-\frac{(ly)^2}{2(\mu+1)(\mu+2)}-\frac{(ly)^3}{3(\mu+1)(\mu+2)(\mu+3)}-\ldots\&c.$$

$$+\frac{\mu}{ly}+\frac{(\mu-1)\mu}{2(ly)^2}+\frac{(\mu-2)(\mu-1)\mu}{3(ly)^3}+\ldots\ldots\&c.$$

ubi m est numerus integer positivus proximus valori $\pm lz$; item μ numerus integer positivus proximus valori $\pm ly$.

Si in hac aequatione (d) introducantur valores $z = x^{\alpha+1}$ $y = x^{\beta+1}$; habebitur valor integralis propositi per series quae sunt functiones ipsius x, & quae convergunt pro iis casibus; quibus divergit series (a).

Pecu-

Peculiarem attentionem meretur aequatio (d) quando fumitur $\alpha = n\sqrt{-1}$; $\beta = -n\sqrt{-1}$; tunc enim habetur $z = x^{1+n\sqrt{-1}} = x e^{nlx\sqrt{-1}} = x \text{ cof. } nlx + x\sqrt{-1} \text{ fin. } nlx$; $lz = (1+n\sqrt{-1})\,lx$; habetur item $y = x^{1-n\sqrt{-1}}$ $x e^{-nlx\sqrt{-1}} = x \text{ cof. } nlx - x\sqrt{-1} \text{ fin. } nlx$; $ly = (1-n\sqrt{-1})\,lx$. Quare habebitur fumpto $m = \mu$

$$z\left(\frac{1}{lz} + \frac{1}{(lz)^2} + 2\frac{1}{(lz)^3} + 2.3\frac{1}{(lz)^4} + \ldots + 2.3.4\ldots(m-1)\frac{1}{(lz)^m}\right)$$
$$-y\left(\frac{1}{ly} + \frac{1}{(ly)^2} + 2\frac{1}{(ly)^3} + 2.3\frac{1}{(ly)^4} + \ldots + 2.3.4\ldots(\mu-1)\frac{1}{(ly)^\mu}\right)$$
$$=$$
$$-x \text{ cof. } nlx \left[\frac{n}{(1+n^2)lx} + \frac{2n}{(1+n^2)^2(lx)^2} + 2\frac{3n-n^3}{(1+n^2)^3(lx)^3} + \ldots + 2.3.4\ldots(m-1)\frac{mn - \frac{m(m-1)(m-2)}{2.3}n^3 + \ldots}{(1+n^2)^m(lx)^m}\right] 2\sqrt{-1}$$
$$+x \text{ fin. } nlx \left[\frac{1}{(1+n^2)lx} + \frac{1-n^2}{(1+n^2)^2(lx)^2} + 2\frac{1-3n^2}{(1+n^2)^3(lx)^3} + \ldots + 2.3.4\ldots(m-1)\frac{1 - \frac{m(m-1)}{2}n^2 + \ldots}{(1+n^2)^m(lx)^m}\right] 2\sqrt{-1}$$

habebitur quoque

$$\left.\begin{array}{l} A - 1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \ldots\ldots - \frac{1}{m} \\ -A + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots\ldots + \frac{1}{\mu} \end{array}\right\} = 0$$

item $+l \pm lz - l \pm ly = l\frac{1+n\sqrt{-1}}{1-n\sqrt{-1}} = 2\sqrt{-1}\text{ A tang. } n.$

Deinde erit

$$\frac{lz}{m+1} + \frac{(lz)^2}{2(m+1)(m+2)} + \frac{(lz)^3}{3(m+1)(m+2)(m+3)} + \ldots \&c. \quad -$$

$$-\frac{ly}{\mu+1}-\frac{(ly)^2}{2(\mu+1)(\mu+2)}-\frac{(ly)^3}{3(\mu+1)(\mu+2)(\mu+3)}-\ldots\ \&c. =$$

$$2\sqrt{-1}\left(\frac{nlx}{m+1}+\frac{2n(lx)^2}{2(m+1)(m+2)}+\frac{(3n-n^3)(lx)^3}{3(m+1)(m+2)(m+3)}+\ldots\&c.\right)$$

ac denique

$$-\frac{m}{lz}-\frac{(m-1)m}{2(lz)^2}-\frac{(m-2)(m-1)m}{3(lz)^3}-\ldots\ldots\ \&c.$$

$$+\frac{\mu}{ly}+\frac{(\mu-1)\mu}{2(ly)^2}+\frac{(\mu-2)(\mu-1)\mu}{3(ly)^3}+\ldots\ldots\ \&c.$$

$$=$$

$$2\sqrt{-1}\left(\frac{mn}{(1+n^2)lx}+\frac{(m-1)m\times 2n}{2(1+n^2)^2(lx)^2}+\frac{(m-2)(m-1)m(3n-n^3)}{3(1+n^2)^3(lx)^3}+\ldots\&c.\right)$$

Quare erit $(e)\ \ldots\ldots \int\frac{dx\ \text{fin.}\ nlx}{lx} =$

$$-x\,\text{cof.}\,nlx\left[\begin{array}{l}\frac{n}{(1+n^2)lx}+\frac{2n}{(1+n^2)^2(lx)^2}+2\frac{3n-n^3}{(1+n^2)^3(lx)^3}+\ldots\ldots \\ \ldots+2.3.4\ldots(m-1)\frac{mn-\frac{m(m-1)(m-2)}{2\cdot 3}n^3+\ldots}{(1+n^2)^m(lx)^m}\end{array}\right]$$

$$+x\,\text{fin.}\,nlx\left[\begin{array}{l}\frac{1}{(1+n^2)lx}+\frac{1-n^2}{(1+n^2)^2(lx)^2}+2\frac{1-3n^2}{(1+n^2)^3(lx)^3}+\ldots\ldots \\ \ldots+2\cdot 3\cdot 4\ldots(m-1)\frac{1-\frac{m(m-1)}{2}n^2+\ldots}{(2+n\ )^m(lx)^m}\end{array}\right]$$

$$+\frac{nlx}{m+1}+\frac{2n(lx)^2}{2(m+1)(m+2)}+\frac{(3n-n^3)(lx)^3}{3(m+1)(m+2)(m+3)}+\ldots\ldots\&c.$$

$$+\frac{mn}{(2+n^2)lx}+\frac{(m-1)m\times 2n}{2(1+n^2)^2(lx)^2}+\frac{(m-2)(m-1)m(3n-n^3)}{3(1+n^2)^3(lx)^3}+\ldots\&c.$$

Cum itaque alterutra ex feriebus (b) & (e) convergat; habebimus per feries valorem formulae integralis $\int\frac{dx\ \text{fin.}\ nlx}{lx}$ pro quocunque valore ipfius x. Hacte-

Hactenus evoluta est formula $\int \frac{(x^\alpha - x^\beta)\,dx}{lx}$, ex qua derivatum est etiam integrale $\int \frac{dx \text{ fin. } nlx}{lx}$. Ut vero haberi possit etiam aliud integrale analogum $\int \frac{dx \text{ cof. } nlx}{lx}$; evolvamus etiam formulam $\int \frac{(x^\alpha + x^\beta)\,dx}{lx}$; quod ut fiat additis simul aequationibus (A), & (B) habebitur

$$(f) \int \frac{(x^\alpha + x^\beta)\,dx}{lx} = 2\,A + \log.(lz\,ly) + lz\,ly + \frac{(lz)^2 + (ly)^2}{2 \cdot 2} + \frac{(lz)^3 + (ly)^3}{2 \cdot 3 \cdot 3} + \ldots$$

$$\text{feu } (g) \ldots\ldots \int \frac{(x^\alpha + x^\beta)\,dx}{lx} =$$

$$2\,A + l(\alpha+1) + l(\beta+1) + (lx)^2 + (\alpha+1)(\beta+1)(lx)^2 + \frac{(\alpha+1)^2 + (\beta+1)^2}{2 \cdot 2}(lx)^2 + \frac{(\alpha+1)^3 + (\beta+1)^3}{2 \cdot 3 \cdot 3}(lx)^3 + \ldots\ldots$$

ubi $A = 0{,}577215\ \ 664901$
numerus supra inventus in prima parte Adnot. pag. 11.

Si series (g) non convergat adeo, ut facile habeatur valor integralis; tunc facto ut supra $x^{\alpha+1} = z$; $x^{\beta+1} = y$, ac proinde $\int \frac{(x^\alpha + x^\beta)\,dx}{lx} = \int \frac{dz}{lz} + \int \frac{dy}{ly}$ habebuntur aliae series convergentes substituendae ex aequat. (10) Adnotat. I. ad hoc Caput pag. 10; erit nempe

$$(h) \ldots \int \frac{(x^\alpha + x^\beta)\,dx}{lx} =$$

$x^{\alpha+1}$

$$x^{\alpha+1}\left[\begin{array}{l}\frac{1}{(\alpha+1)lx}+\frac{1}{(\alpha+1)^2(lx)^2}+2\frac{1}{(\alpha+1)^3(lx)^3}+2.3\frac{1}{(\alpha+1)^4(lx)^4}\\ +\ldots\ldots+2.3.4\ldots(m-1)\frac{1}{(\alpha+1)^m(lx)^m}\end{array}\right],$$

$$+A-1-\frac{1}{2}-\frac{1}{3}-\ldots\ldots-\frac{1}{m}+l\pm(\alpha+1)lx$$

$$+\frac{(\alpha+1)lx}{m+1}+\frac{(\alpha+1)^2(lx)^2}{2(m+1)(m+2)}+\frac{(\alpha+1)^3(lx)^3}{3(m+1)(m+2)(m+3)}+\ldots$$

$$-\frac{m}{(\alpha+1)lx}-\frac{(m-1)m}{2(\alpha+1)^2(lx)^2}-\frac{(m-2)(m-1)m}{3(\alpha+1)^3(lx)^3}-\ldots\ldots$$

$$+x^{\beta+1}\left[\begin{array}{l}\frac{1}{(\beta+1)lx}+\frac{1}{(\beta+1)^2(lx)^2}+2\frac{1}{(\beta+1)^3(lx)^3}+2.3\frac{1}{(\beta+1)^4(lx)^4}\\ +\ldots\ldots+2.3.4\ldots(\mu-1)\frac{1}{(\beta+1)^\mu(lx)^\mu}\end{array}\right],$$

$$+A-1-\frac{1}{2}-\frac{1}{3}-\ldots\ldots-\frac{1}{\mu}+l\pm(\beta+1)lx$$

$$+\frac{(\beta+1)lx}{\mu+1}+\frac{(\beta+1)^2(lx)^2}{2(\mu+1)(\mu+2)}+\frac{(\beta+1)^3(lx)^3}{3(\mu+1)(\mu+2)(\mu+3)}+\ldots\ldots$$

$$-\frac{\mu}{(\beta+1)lx}+\frac{(\mu-1)\mu}{2(\beta+1)^2(lx)^2}-\frac{(\mu-2)(\mu-1)\mu}{3(\beta+1)^3(lx)^3}-\ldots\ldots$$

Si in aequatione (g) ponamus $\alpha=n\sqrt{-1}$; $\beta=-n\sqrt{-1}$; habebimus $x^\alpha+x^\beta=x^{n\sqrt{-1}}+x^{-n\sqrt{-1}}=e^{nlx\sqrt{-1}}+e^{-nlx\sqrt{-1}}=2\,\text{cos}.\,nlx$; unde emerget aequatio

$$(k)\ldots\ldots\int\frac{dx\ \text{cos}.\ nlx}{lx}=$$

$$A+\frac{1}{2}l(1+n^2)+\frac{1}{2}(lx)^2+\frac{1}{2}(1+n^2)(lx)^2+\frac{1-n^2}{2.2}(lx)^2$$

$$+\frac{1-3n^2}{2.3.3}(lx)^3+\ldots\ldots$$

in

in qua aequatione si ponatur $x=0$; erit

$$\int \frac{dx \cos.\, nlx}{lx} = A + \frac{1}{2}\, l(1+n^2)$$

Si series superior (k) non inserviat ad habendum proxime valorem integralis ob defectum convergentiae; tunc substitutis in serie (h) valoribus $x^{1+n\sqrt{-1}} = xe^{nlx\sqrt{-1}} = x\cos.\, nlx + x\sqrt{-1}\,\sin.\, nlx$; item $x^{1-n\sqrt{-1}} = xe^{-nlx\sqrt{-1}} = x\cos.\, nlx - x\sqrt{-1}\,\sin.\, nlx$, ac sumpto $m=\mu$; erit

$$(l) \ldots\ldots \int \frac{dx \cos.\, nlx}{lx} =$$

$$x\cos.nlx \left[\frac{1}{(1+n^2)lx} + \frac{1-n^2}{(1+n^2)^2(lx)^2} + 2\,\frac{1-3n^2}{(1+n^2)^3(lx)^3} + \ldots\ldots + 2.3.4\ldots(m-1)\,\frac{1-\frac{m(m-1)}{2}n^2+\frac{m(m-1)(m-2)(m-3)}{2\cdot3\cdot4}n^4\ldots}{(1+n^2)^m(lx)^m} \right]$$

$$+\,x\sin.nlx \left[\frac{n}{(1+n^2)lx} + \frac{2n}{(1+n^2)^2(lx)^2} + 2\,\frac{3n-n^3}{(1+n^2)^3(lx)^3} + \ldots\ldots + 2.3.4\ldots(m-1)\,\frac{mn-\frac{m(m-1)(m-2)}{2\cdot3}n^3+\ldots}{(1+n^2)^m(lx)^m} \right]$$

$$+A-1-\frac{1}{2}-\frac{1}{3}-\frac{1}{4}-\ldots\ldots-\frac{1}{m}+\frac{1}{2}\,l(1+n^2)(lx)^2$$

$$+\frac{lx}{m+1}+\frac{(1-n^2)(lx)^2}{2(m+1)(m+2)}+\frac{(1-3n^2)(lx)^3}{3(m+1)(m+2)(m+3)}+\ldots\ldots$$

$$-\frac{m}{(1+n^2)lx}-\frac{(m-1)m(1-n^2)}{2(1+n^2)^2(lx)^2}-\frac{(m-2)(m-1)m(1-3n^2)}{3(1+n^2)^3(lx)^3}-\ldots\ldots$$

Harum itaque aequationum (k), & (l) alterutra exhibebit series convergentes pro valore integralis

$$\int \frac{dx \cos.\, nlx}{lx}$$

Atque his omnibus absoluta est evolutio formulae integralis

$$\int \frac{(x^{\alpha} \pm x^{\beta})\, dx}{l x}$$

Evolutio completa formulae integralis

$$\int dx \left(\frac{1}{1-x} + \frac{1}{lx} \right)$$

IN Tom. IV. Nov. Act. Acad. Scient. Petrop. ad annum 1786. invenitur pag. 3. Commentarius Euleri, cui titulus: *Evolutio formulae integralis* $\int dx \left(\frac{1}{1-x} + \frac{1}{lx} \right)$ *a termino* $x = 0$, *usque ad* $x = 1$ *extensae*; qui sic incipit: *ista formula integralis eo magis est notatu digna, quod ejus valorem ostendi convenire cum eo, quem praebet ista expressio*: $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots\ldots + \frac{1}{n} - l\, n$, *si numerus* n *sumatur infinite magnus, & quem per approximationem olim* (in Calculo Differ. Part. poster. Cap. VI.) *inveni esse* 0, 5772156649015325, *cujus valorem nullo adhuc modo ad mensuras transcendentes jam cognitas redigere potui; unde haud inutile erit resolutionem hujus formulae propositae pluribus modis tentare*. Id autem praestat modis quinque, quibus varias approximationes obtinet; per quartum vero habet

$$\int dx \left(\frac{1}{1-x} + \frac{1}{lx} \right) =$$

$$\frac{1}{2}\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \&c. \right)$$

$$- \frac{1}{3}\left(1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} + \&c. \right)$$

$+$

$$+\frac{1}{4}\left(1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\frac{1}{5^4}+\&c.\right)$$
$$-\frac{1}{5}\left(1+\frac{1}{2^5}+\frac{1}{3^5}+\frac{1}{4^5}+\frac{1}{5^5}+\&c.\right)$$
$$+\&c.$$

cuius expressionis ostendit *valorem esse numerum* illum memorabilem 0, 5772156649015325 (V. Differt. De numero memorabili in summatione progressionis harmonicae naturalis occurrente. Acta Acad. pro anno 1781. Pars posterior pag. 49. seqq.).

Evolutio completa formulae propositae habebitur, si pro quocumque valore x habeamus series convergentes, quibus exhibeatur valor integralis. Cum vero sit $\int\frac{dx}{1-x}=l\frac{1}{1-x}$, & $\int\frac{dx}{lx}$ jam habeamus ex Adnot. I. ad hoc Caput IV.; in promptu erunt duae series exhibentes valorem integralis

$$\int dx\left(\frac{1}{1-x}+\frac{1}{lx}\right)=$$

$$(a)\ldots\text{A}+l\frac{\pm lx}{1-x}+lx+\frac{(lx)^2}{2.2}+\frac{(lx)^3}{2.3.3}+\frac{(lx)^4}{2.3.4.4}+\ldots$$

$$(b)\ldots\text{A}-1-\frac{1}{2}-\frac{1}{3}-\frac{1}{4}-\ldots-\frac{1}{n}+l\frac{\pm lx}{1-x}$$
$$+x\left(\frac{1}{lx}+\frac{1}{(lx)^2}+2\frac{1}{(lx)^3}+2.3\frac{1}{(lx)^4}+\ldots 2.3.4\ldots(n-1)\frac{1}{(lx)^n}\right)$$
$$+\frac{lx}{n+1}+\frac{(lx)^2}{2(n+1)(n+2)}+\frac{(lx)^3}{3(n+1)(n+2)(n+3)}+\ldots\ldots$$
$$-\frac{n}{lx}-\frac{(n-1)n}{2(lx)^2}-\frac{(n-2)(n-1)n}{3(lx)^3}-\ldots\ldots$$

ubi A est numerus supra positus.

Si lx sit quantitas proxima unitati positivae, vel negativae; adhibenda erit series (a) utpote convergens. Si vero

vero lx superet admodum unitatem positivam, vel negativam; tunc sumpto n numero positivo proxime aequali ipsi $\pm lx$, adhibebitur series (b). Signa vero in formula $l\frac{\pm lx}{1-x}$ ea assumentur, per quae $\frac{\pm lx}{1-x}$ sit quantitas positiva, ut supra demonstravimus.

Fiat nunc $x = 1 - \omega$ sumpta pro ω quantitate infinite parva; erit $lx = -\omega - \frac{\omega^2}{2} - \frac{\omega^3}{3} - \&c.$ sive $lx = -\omega$; ac proinde $l\frac{-lx}{1-x} = l\frac{\omega}{\omega} = 0$. Quare casu quo sumitur $x = 1$, habebitur ex aequatione (a) $\int dx \left(\frac{1}{1-x} + \frac{1}{lx}\right) = A = 0,577215 \ldots\ldots$ uti Eulerus invenit.

Notanda vero est curva, per cuius quadraturam Eulerus exhibuit evolutionem primam geometricam, quae est ex earum genere, quae ex puncto aliquo incipiunt subito veluti ex abrupto. Cuius proprietatis observandae gratia ipsam evolutionem primam posuit, quamvis ex ea difficile posset haurire valorem formulae. Quam ita concludit: *sufficit formam huius curvae prorsus singularis quippe quae in puncto* C *subito incipit expendisse*. Apparet autem statim haec proprietas *considerando lineam curvam*, *cuius abscissae* x *respondeat applicata* $y = \frac{1}{1-x} + \frac{1}{lx}$. Nam *primo quidem evidens est hanc curvam neutiquam in regionem abscissarum negativarum porrigi*, *sed a termino* $x = 0$ *incipere*. *Posito autem* $x = 0$ *manifesto fit* $y = 1$ *ob* $lx = \infty$. Cum vero cuicumque x positivae non respondeat nisi una ordinata y; manifestum est curvam non habere nisi unicum ramum incipientem ex eo puncto ubi habetur $x = 0$, & ordinata finita $y = 1$.

Appen-

Appendix ad Adnotationem I.

IN doctrina logarithmorum consultum est per series, ut dato numero inveniri possit ejus logarithmus, ac vicissim dato logarithmo inveniri possit numerus, cujus logarithmus est. Nos integrale hujusmodi $\int \frac{dz}{lz}$, quod appellavimus hyperlogarithmum z, atque hoc modo indicavimus $l'z$, ut esset $\int \frac{dz}{lz} = l'z$ ita assignavimus per series, ut quicumque esset valor ipsius lz haberi posset valor $l'z$. Ad absolutionem doctrinae requiri videtur, ut habeantur aequationes, quibus viceversa dato $l'z$ inveniri possit lz praesertim cum aequationibus jam inventis per series applicari non possit methodus regressus serierum. Ut ergo etiam hanc alteram partem conficiamus sit $\frac{dz}{lz} = du$; adeo ut sit $\int \frac{dz}{lz} = l'z = H + u$; erit $dz = du\,lz$; ac facto du constante; erit $z\,ddz = du\,dz$. Sit nunc

$$z = K + Au + Bu^2 + Cu^3 + Du^4 + Eu^5 + Fu^6 + Gu^7 + \ldots$$

$$dz = du(A + 2Bu + 3Cu^2 + 4Du^3 + 5Eu^4 + 6Fu^5 + 7Gu^6 + \ldots)$$

$$ddz = du^2(2B + 2.3Cu + 3.4Du^2 + 4.5Eu^3 + 5.6Fu^4 + 6.7Gu^5 + \ldots)$$

$$\frac{zddz}{du^2} = \left\{\begin{array}{llllll} 1.2KB & +2.3KCu & +3.4KDu^2 & +4.5KEu^3 & +5.6KFu^4 & +\ldots \\ & +1.2AB & +2.3AC & +3.4AD & +4.5AE & +\ldots \\ & & +1.2BB & +2.3BC & +3.4BD & +\ldots \\ & & & +1.2CB & +2.3CC & +\ldots \\ & & & & +1.2DB & +\ldots \end{array}\right.$$

$$= \frac{dz}{du} = A + 2Bu + 3Cu^2 + 4Du^3 + 5Eu^4 + \ldots$$

unde

unde oriuntur aequationes

$$B = \frac{A}{1.2\,K}$$

$$C = \frac{2(1-A)B}{2.3\,K}$$

$$D = \frac{3(1-2A)C - 1.2\,BB}{3.4\,K}$$

$$E = \frac{4(1-3A)D - (1.2+2.3)BC}{4.5\,K}$$

$$F = \frac{5(1-4A)E - (1.2+3.4)BD - 2.3\,CC}{5.6\,K}$$

$$G = \frac{6(1-5A)F - (1.2+4.5)BE - (2.3+3.4)CD}{6.7\,K}$$

&c.

unde satis apparet lex etiam pro aliis aequationibus successivis in infinitum, quibus coefficientes determinantur.

Cum pro integrali completo aequationis secundi gradus

$$z\,ddz = du\,dz$$

sint duae constantes indeterminatae A, & K; ut eas determinemus ratione commoda, in primis observandum est quod si fiat $u = 0$; erit $z = K$; $lz = lK$; $l'z = H$. Cum K non possit sumi $= 0$, ne coefficientes B, C, D &c. fiant infiniti, & cum praestet sumere K majorem unitate ut series exprimens valorem z convergat ratione coefficientium B, C, &c. commodum erit sumere $K = e$ basi logarithmorum hyperbolicorum, quo posito est $lK = 1$, ac propterea

$$\int \frac{dz}{lz} = l'z = a + l + \frac{1}{2.2} + \frac{1}{2.3.3} + \frac{1}{2.3.4.4} + \frac{1}{2.3.4.5.5} + \&c. = H$$

ubi $a = 0,577215$ &c. ac propterea $H = 1,895117$ 816355 936755 678109

Ut

Ut nunc determinetur etiam A; sumatur $u = \omega$ quantitati infinitesimae, ut sit $z = e + A\omega$; $lz = le \left(1 + \frac{A\omega}{e}\right) = 1 + \frac{A\omega}{e}$, quo posito habetur

$$l'z = a + 1 + \frac{1}{2.2} + \frac{1}{2.3.3} + \frac{1}{2.3.4.4} + \dots\dots$$
$$+ \frac{A\omega}{e}\left(1 + \frac{1}{2} + \frac{1}{2.3} + \frac{1}{2.3.4} + \dots\dots\right)$$
$$= H + A\omega =$$

Quare cum sit etiam $l'z = H + u$ seu $H + \omega$ erit $A = 1$. Quare tandem

$$z = e + u + \frac{1}{2e}u^2 - \frac{1}{2.3.4e^3}u^4 + \frac{2}{2.3.4.5e^4}u^5$$
$$+ \frac{1}{2.3.4.5.6e^5}u^6 - \frac{18}{2.3.4.5.6.7e^6}u^7 \text{ \&c.}$$

Si u sumatur major unitate, multo magis vero si sit $7e$; parum converget series, quam nuper exhibuimus. Quare tunc alio modo determinanda erit constans H. Quod ut fiat modo satis generali, ut tuti esse possimus de convergentia seriei exhibentis valorem z pro quocumque valore u; sumatur $K = e^m$, ut sit $lK = m$; ac propterea sumpto $u = 0$; $z = K$

$$l'z = a + lm + m + \frac{m^2}{2.2} + \frac{m^3}{2.3.3} + \frac{m^4}{2.3.4.4} + \dots\dots = H$$

vel sumpta aequatione (10)

$$l'z = e^m\left(\frac{1}{m} + \frac{1}{m^2} + 2\frac{1}{m^3} + 2.3\frac{1}{m^4} + \dots 2.3.4\dots(n-1)\frac{1}{m^n}\right)$$
$$+ a - 1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} \dots\dots - \frac{1}{n} + lm$$
$$+ \frac{m}{n+1} + \frac{m^2}{2(n+1)(n+2)} + \frac{m^3}{3(n+1)(n+2)(n+3)} + \dots$$

$$-\frac{n}{m}-\frac{n(n-1)}{2m^{2}}-\frac{n(n-1)(n-2)}{3m^{3}}-\ldots\ldots=\mathrm{H}$$

ubi n eſt numerus integer proximus valori ipſius m.

Hoc modo determinato H per K ut determinetur A ſumpto $u=\omega$ quantitate infiniteſima, ut ſit $z=e^{m}+\mathrm{A}\omega$; $lz=m+l\left(1+\frac{\mathrm{A}\omega}{e^{m}}\right)-m+\frac{\mathrm{A}\omega}{e^{m}}$; erit

$$l'z=\begin{matrix} a+lm+m+\frac{m^{2}}{2.2}+\frac{m^{3}}{2.3.3}+\&c. \\ +\frac{\mathrm{A}\omega}{me^{m}}\left(1+m+\frac{m^{2}}{2}+\frac{m^{3}}{2.3}\right.+\&c. \end{matrix}$$

$$=\mathrm{H}+\frac{\mathrm{A}\omega}{m}.$$

Sed eſt $l'z=\mathrm{H}+\omega$; ergo $\mathrm{A}=m$; ex quo orietur nova ſeries $z=e^{m}+mu+\frac{m}{1.2e^{m}}u^{2}+\frac{(1-m)m}{2.3e^{2m}}u^{3}+\&c.$

Ut hnius ſeriei adhibendae methodus, atque utilitas appareat; notemus aliquos valores praecipuos formulae integralis $\int\frac{dz}{lz}$, quae annihilatur poſito $z=0$

Si ſit $z=e^{-\infty}$	$=0$	erit $\int\frac{dz}{lz}$	$=0$
$=e^{-1}$	$=0,36788$		$=-0,219383$
$=e^{0}$	$=1$		$=-\infty$
$=e^{\frac{1}{e}}$	$=1,44467$		$=-0,01812$
$=e$	$=2,71828$		$=+1,89511$

Ex quo apparet pro aliquo valore z medio inter $e^{\frac{1}{e}}$, & e^{1} rurſus fore $\int\frac{dz}{lz}=l'z=0$. Hic autem ita inveniri poteſt ope ſuperioris aequationis. Quoniam ſumpto

$z=$

$z = K = e$, fit $l'z = +1,89 = H$, ut fit $H + u = 0$; fumi debet $u = -1,89$, qui valor cum praefcindendo a figno negativo fit maior unitate, finit parum convergere feriem $z = e + u + \frac{1}{2e} u^2 + \&c.$ Sumatur ergo $z = K = e^{\frac{1}{e}}$; quo pofito habetur $l'z = -0,01812 = H$; ac proinde aequatio $H + u = 0$ dat $u = 0,01812$; cumque fit $m = \frac{1}{e}$; habetur $z = e^{\frac{1}{e}} + \frac{0,01812}{e} + \frac{(0,01812)^2}{2 . e\, e^{\frac{1}{e}}} + \frac{(0,01812)^3 (e-1)}{2 . 3\, e^2 e^{\frac{2}{e}}} + \&c.$, feu $z = 1,45137$.

Evolutio completa formulae integralis

$$\int x^{f-1} dx (lx)^{\frac{m}{n}}$$

In novis Commentariis Academiae Scientiarum Petropolitanae Tom. XVI. ann. 1771. Eulerus Commentarium inferuit, cui titulus: *Evolutio formulae integralis* $\int x^{f-1} dx (lx)^{\frac{m}{n}}$ *integratione a valore* $x = 0$ *ad valorem* $x = 1$ *extenfa*. Hic vero requiretur valor hujus formulae integralis pro quocumque valore ipfius x, qui non reddat ipfam formulam imaginariam.

Atque in primis cum a plerifque excludantur logarithmi quantitatum negativarum; excludemus & nos omnes valores negativos ipfius x.

Secundo: cum $\frac{m}{n}$ fit fractio reducta ad minimos terminos; fi fit n numerus par; pro valoribus pofitivis ipfius

 x uni-

x unitate minoribus, atque adeo etiam ad ipſum limitem $x=0$, erit lx quantitas negativa; ideoque $(lx)^{\frac{m}{n}}$ erit quantitas imaginaria. Si ergo ſit n numerus par; excludemus omnes valores ipſius x unitate minores. Si n ſit numerus impar; nullos valores poſitivos ipſius x excludemus.

Tertio cum ad formulam $\int x^{f-1} dx (lx)^{\frac{m}{n}}$ referatur etiam formula $\int x^{f-1} dx \left(l\frac{1}{x}\right)^{\frac{m}{n}} = \int x^{f-1} dx \left(-lx\right)^{\frac{m}{n}}$; quam praeſertim conſiderat Eulerus in problemate ſexto generali citati Commentarii; ſi n ſit numerus par; pro valoribus ipſius x unitate majoribus uſque in infinitum erit $\left(-lx\right)^{\frac{m}{n}}$ quantitas imaginaria. Si ergo ſit n numerus par; excludemus in ſecunda formula $\int x^{f-1} dx \left(l\frac{1}{x}\right)^{\frac{m}{n}}$ omnes valores x unitate majores. Si n ſit impar; nullos excludemus.

His animadverſis evolutio formularum (quas jam ut duas ſemper conſiderabimus, licet ſecunda referatur ad primam propoſitam) erit completa ſi exhibeamus integrale per ſéries convergentes pro quocumque valore x non excluſo.

Quod attinet ad conſtantem ingreſſam per integrationem; ea quoties licebit accipietur talis, ut integrale evaneſcat poſito $x=0$. Haec ergo conditio ſemper adhibenda erit pro ſecunda formula $\int x^{f-1} dx \left(l\frac{1}{x}\right)^{\frac{m}{n}}$. Pro prima formula vero $\int x^{f-1} dx \left(lx\right)^{\frac{m}{n}}$ adhuc ea conditio adhibebitur ſi fuerit n numerus impar. Si vero n fuerit par; cum eo caſu non poſſit ſumi $x=0$, quin $lx=-\infty$ reddat imaginariam quantitatem $\left(lx\right)^{\frac{m}{n}}$, atque adeo ipſam formu-

lam

lam integralem; praeſtabit adhibere conditionem conſtantis ut integrale evaneſcat poſito $x = 1$, qui valor eſt minimus poſſibilis, uti $x = 0$ erat initium valorum poſſibilium in caſu praecedenti.

Jam ergo evolvamus primam formulam propoſitam ubi n ſit numerus impar.

Fiat $lx = \frac{y}{f}$; $\frac{m}{n} = \nu$; erit $x = e^{\frac{y}{f}}$; $dx = e^{\frac{y}{f}} \frac{dy}{f}$

$\int x^{f-1} dx\,(lx)^{\frac{m}{n}} = \frac{1}{f^{\nu+1}} \int e^{y} y^{\nu} dy$. Eſt vero

$\int e^{y} y^{\nu} dy = \int y^{\nu} dy \left(1 + y + \frac{y^2}{2} + \frac{y^3}{2 \cdot 3} + \frac{y^4}{2 \cdot 3 \cdot 4} + \ldots\right)$

ac proinde

$$(1) \ldots \int e^{y} y^{\nu} dy = C + \frac{1}{\nu+1} y^{\nu+1} + \frac{1}{\nu+2} y^{\nu+2} + \frac{1}{\nu+3} \cdot \frac{y^{\nu+3}}{2} + \frac{1}{\nu+4} \cdot \frac{y^{\nu+4}}{2 \cdot 3} + \ldots\ldots;$$

Rurſus

$$(2) \ldots \int e^{y} y^{\nu} dy = y^{\nu} e^{y} - \int \nu y^{\nu-1} e^{y} dy$$

$$(3) \quad = y^{\nu} e^{y} - \nu y^{\nu-1} e^{y} + \int \nu(\nu-1) y^{\nu-2} e^{y} dy$$

$$(4) \quad = y^{\nu} e^{y} - \nu y^{\nu-1} e^{y} + \nu(\nu-1) y^{\nu-2} e^{y} - \int \nu(\nu-1)(\nu-2) y^{\nu-3} e^{y} dy$$

$$(5) \quad = y^{\nu} e^{y} - \nu y^{\nu-1} e^{y} + \nu(\nu-1) y^{\nu-2} e^{y} - \nu(\nu-1)(\nu-2) y^{\nu-3} e^{y} + \ldots\ldots \pm \nu(\nu-1)(\nu-2) \ldots\ldots (\nu-\mu) y^{\nu-\mu-1} e^{y} \mp \int \nu(\nu-1)(\nu-2) \ldots\ldots (\nu-\mu-1) y^{\nu-\mu-2} e^{y} dy$$

Quaeratur nunc pro aequatione (1) conſtans C talis ut integrale evaneſcat poſito $y = -\infty$, ſeu $x = 0$. Evolvatur per ſubſtitutionem $e^{y} = 1 + y + \frac{y^2}{2} + \ldots$. ſecundus terminus ſummatorius in aequatione (2); habebitur

$$(6) \int e^{y} y^{\nu} dy = y^{\nu} e^{y} + D - y^{\nu} - \frac{\nu}{\nu+1} y^{\nu+1} - \frac{\nu}{\nu+2} \cdot \frac{y^{\nu+2}}{2} - \ldots$$

ubi

ubi D ſit conſtans adjicienda ut integrale evaneſcat ſumpto $y = -\infty$. Evolvatur in hac ſerie (6) terminus $y^\nu e^y$, & facta reductione cum terminis ſequentibus poſt conſtantem D conferatur ſeries inde enata cum ſerie aequationis (1); invenietur $D = C$.

Eodem modo evoluto ſecundo termino ſummatorio aequationis (3) habebitur

$$(7) \ldots \int e^y y^\nu dy = y^\nu e^y - \nu y^{\nu-1} e^y + E + \nu y^{\nu-1} + (\nu-1) y^\nu + \ldots$$

ubi E ſit conſtans ejuſdem legis. In hac aequatione (7) ſi evolvatur terminus $-\nu y^{\nu-1} e^y$, & facta reductione in ſecundo membro aequationis cum terminis poſitis poſt conſtantem E conferatur ſeries enata cum ſerie aequationis (6); invenietur $E = D = C$. Hac methodo perpetuo progrediendo ſi evolvatur ſecundus terminus ſummatorius aequationis (5) ita ut ſit

$$\begin{aligned}(8) \ldots \int e^y y^\nu dy = {} & y^\nu e^y - \nu y^{\nu-1} e^y + \nu(\nu-1) y^{\nu-2} e^y - \ldots \\ & \pm \nu(\nu-1)(\nu-2)\ldots(\nu-\mu)\, y^{\nu-\mu-1} e^y + K \\ & \mp \nu(\nu-1)(\nu-2)\ldots(\nu-\mu-1)\Big[\frac{1}{\nu-\mu-1} y^{\nu-\mu-1} \\ & + \frac{1}{\nu-\mu} y^{\nu-\mu} + \frac{1}{\nu-\mu+1} \cdot \frac{y^{\nu-\mu+1}}{2} \ldots \\ & + \frac{1}{\nu-\mu+\rho-1} \cdot \frac{y^{\nu-\mu+\rho-1}}{2 \cdot 3 \cdot 4 \ldots \rho} + \ldots\Big]\end{aligned}$$

K vero ſit conſtans adjicienda ut integrale evaneſcat ſumpto $y = -\infty$, quicumque ſit numerus μ; invenietur ſemper $K = C$. Jam ergo inquiratur in hac ſerie generali valor ipſius K.

Sumpto $y = -\infty$ terminus $y^\nu e^y$ fit infinite parvus. Nam logarithmus termini $-y^\nu e^y$ eſt $\nu \log. -y + y \log. e = \nu l \infty - \infty = -\infty$, cum $l\,\infty$ negligi poſſit prae ipſo ∞. Ergo terminus $-y^\nu e^y$ eſt infiniteſimus, ac proinde etiam $y^\nu e^y$.

Jam

Jam vero termini ſequentes in ſerie ante conſtantem K poſt ipſum terminum $y^{\nu} e^{y}$ convergunt donec $\frac{\nu-\mu}{-y}$, qui eſt factor generalis termini ſequentis ſit fractio; quare ſi non ſumatur $\mu > y$; tota ſeries poſita ante conſtantem K evaneſcit.

Cum in ſigno duplici $\mp$ poſito poſt conſtantem K ſumenda ſit pars ſuperior — ſi μ ſit impar; contra vero inferior + ſi ſit par; facile apparet formulam

$$K \mp \nu(\nu-1)(\nu-2)\ldots\ldots(\nu-\mu-1)\,[\,\ldots\ldots\,]$$

ita exprimi poſſe

$$K - \nu(1-\nu)(2-\nu)\ldots\ldots(\mu+1-\nu)\,[\,\ldots\ldots\,]$$

ſublata aequivocatione ſigni; quae formula pro caſu $y = -\infty$ debet annihilari.

Eo itaque caſu erit

$$(9)\ \ldots\ K = \nu(1-\nu)(2-\nu)\ldots(\mu+1-\nu)\Big[\frac{1}{\nu-\mu-1}\,y^{\nu-\mu-1} + \frac{1}{\nu-\mu}\,y^{\nu-\mu} + \frac{1}{\nu-\mu+1}\cdot\frac{y^{\nu-\mu+1}}{2} + \ldots + \frac{1}{\nu-\mu+\rho-1}\cdot\frac{y^{\nu-\mu+\rho-1}}{2\cdot3\cdot4\ldots\rho} + \ldots\Big]$$

in qua ſerie terminus generalis eſt

$$\frac{\nu(1-\nu)(2-\nu)\ldots\ldots(\mu+1-\nu)}{2\cdot3\cdot4\ldots\ldots\rho} : \frac{y^{\nu-\mu+\rho-1}}{\nu-\mu+\rho-1}$$

Sit $\mu+1=\rho=-y$; terminus generalis abibit in ſequentem $\frac{1-\nu}{1}\cdot\frac{2-\nu}{2}\cdot\frac{3-\nu}{3}\ldots\ldots\frac{\rho-\nu}{\rho}(-\rho)^{\nu} =$

—

$$-\frac{1-\nu}{1}\cdot\frac{2-\nu}{2}\cdot\frac{3-\nu}{3}\cdots\cdots\frac{\rho-\nu}{\rho}\,\rho^{\nu}\text{ , qui fiat }=-\mathrm{T}$$

(Vide pag. 58. partis primae Adnot.). Series vero sequens erit

$$-\left(\frac{\nu}{\nu+1}\cdot\frac{y}{\rho+1}\mathrm{T}+\frac{\nu}{\nu+2}\cdot\frac{y^2}{(\rho+1)(\rho+2)}\mathrm{T}+\frac{\nu}{\nu+3}\cdot\frac{y^3}{(\rho+1)(\rho+2)(\rho+3)}\mathrm{T}+\ldots\right)$$

Antecedens vero retrocedendo ab ipso termino T erit

$$-\left(\frac{\nu}{\nu-1}\cdot\frac{\rho}{y}\mathrm{T}+\frac{\nu}{\nu-2}\cdot\frac{\rho(\rho-1)}{y^2}\mathrm{T}+\frac{\nu}{\nu-3}\cdot\frac{\rho(\rho-1)(\rho-2)}{y^3}\mathrm{T}+\ldots\right)$$

Quare cum sit ratione infiniti $\frac{y}{\rho+1}=-1$;

$\frac{y^2}{(\rho+1)(\rho+2)}=+1$; $\frac{y^3}{(\rho+1)(\rho+2)(\rho+3)}=-1$, &c.
erit tandem

$$\mathrm{K}=-\mathrm{T}\left[\begin{array}{l}1-\frac{\nu}{\nu+1}+\frac{\nu}{\nu+2}-\frac{\nu}{\nu+3}+\ldots\ldots\\ -\frac{\nu}{\nu-1}+\frac{\nu}{\nu-2}-\frac{\nu}{\nu-3}+\ldots\ldots\end{array}\right]$$

seu $\mathrm{K}=-\nu\mathrm{T}\left(\frac{1}{\nu}+\frac{2\nu}{1-\nu\nu}-\frac{2\nu}{4-\nu\nu}+\frac{2\nu}{9-\nu\nu}-\ldots\right)$

$$=-\frac{m}{n}\mathrm{T}\left(\frac{n}{m}+\frac{2nm}{nn-mm}-\frac{2nm}{4nn-mm}+\frac{2nm}{9nn-mm}-\ldots\right)$$

$$=-\mathrm{T}\,\frac{\frac{m}{n}\pi}{\sin\frac{m}{n}\pi}$$

(Eulerus Introduct. in Analys. inf. Lib. I. Cap. X.)

Inven-

Inventa hoc modo quantitate conſtanti $K = C$; jam habebimus ſubſtitutis valoribus $y = f\,lx$; $= \frac{m}{n}$ in aequatione (1)

$$(10) \ldots \int x^{f-1}\, dx\,(lx)^{\frac{m}{n}} = - \frac{T}{f^{\frac{m}{n}+1}} \cdot \frac{\frac{m}{n}\pi}{\text{ſin}\,\frac{m}{n}\pi} +$$

$$n\,(lx)^{\frac{m}{n}+1} \left[\frac{1}{n+m} + \frac{1}{2n+m}\, f\,lx + \frac{1}{2} \cdot \frac{1}{3n+m}(f\,lx)^2 + \frac{1}{2.3} \cdot \frac{1}{4n+m}(f\,lx)^3 + \&c. \right]$$

ubi eſt $T = \frac{n-m}{n} \cdot \frac{2n-m}{2n} \cdot \frac{3n-m}{3n} \ldots \frac{\rho n - m}{\rho n}\, \rho^{\frac{m}{n}}$.

facto $\rho = \infty$. Patet autem haberi per approximationem valorem ipſius T eo accuratius, quo major ſumitur inter finitos numerus ρ.

Ponatur nunc $x = 1$; erit $f\,lx = 0$; ac niſi fuerit $\frac{m}{n} + 1$ quantitas negativa; erit pro ipſo caſu $x = 1$

$$\int x^{f-1}\, dx\,(lx)^{\frac{m}{n}} = - \frac{T}{f^{\frac{m}{n}+1}} : \frac{\frac{m}{n}\pi}{\text{ſin}\,\frac{m}{n}\pi}$$

poſito nempe quod ſit n numerus impar; atque ipſum integrale annihiletur pro caſu $x = 0$.

Series (10) convergit quotieſcumque $f\,lx$ eſt quantitas fracta; ſi vero $f\,lx$ jam contineat unitates; ſeries eo minus fit opportuna ad habendum valorem integralis, quo plures unitates poſitivae, vel negativae continentur in valore $f\,lx$. Tunc ergo adhibenda erit ſeries (8) praeparata ut ſequitur.

Suma-

Sumatur $\mu + 1 = \rho$ numerus integer positivus proximior valori $\pm y$. Terminus generalis seriei positae post constantem K, qui est ut docuimus

$$\frac{\nu(1-\nu)(2-\nu)\ldots\ldots(\rho-\nu)}{2\cdot 3\cdot 4\ldots\ldots\rho} \quad \frac{y^{\nu-\mu+\rho-1}}{\nu-\mu+\rho-1}$$

fiet $\frac{(1-\nu)(2-\nu)(3-\nu)\ldots\ldots(\rho-\nu)}{1\cdot 2\cdot 3\ldots\ldots\rho}\, y^{\nu}$. Hic terminus fiat $= R$; erit series sequens ad dexteram

$$R\,\frac{\nu}{\nu+1}\cdot\frac{y}{\rho+1} + R\,\frac{\nu}{\nu+2}\cdot\frac{y^2}{(\rho+1)(\rho+2)}$$

$$+ R\,\frac{\nu}{\nu+3} : \frac{y^3}{(\rho+1)(\rho+2)(\rho+3)} + \&c.$$

Series vero antecedens retrocedendo ad sinistram erit

$$\frac{\nu}{\nu-1}\cdot\frac{\rho}{y}\,R + \frac{\nu}{\nu-2}\cdot\frac{\rho(\rho-1)}{y^2}\,R + \frac{\nu}{\nu-3}\cdot\frac{\rho(\rho-1)(\rho-2)}{y^3}\,R + \&c.$$

quae ambae series cum sit $\pm y$ quamproxime $= \mu$; convergunt; secunda vero constat terminis numero finitis. Sumpto ergo

$$R = \frac{n-m}{n}\cdot\frac{2n-m}{2n}\cdot\frac{3n-m}{3n}\ldots\ldots\frac{\rho n-m}{\rho n}(flx)^{\frac{m}{n}}$$

ob $\int x^{f-1}\,dx = \frac{1}{f^{\frac{m}{n}+1}}\int e^y y^{\nu}\,dy$; erit ex aequat. (8)

$$(11) \; \ldots\ldots \int x^{f-1}\,dx\,(lx)^{\frac{m}{n}} =$$

$$\frac{(lx)^{\frac{m}{n}}x^f}{f}\left[\begin{array}{l} 1-\frac{m}{n}\cdot\frac{1}{flx}+\frac{m}{n}\left(\frac{m}{n}-1\right)\frac{1}{(flx)^2}-\frac{m}{n}\left(\frac{m}{n}-1\right)\left(\frac{m}{n}-2\right)\frac{1}{(flx)^3} \\ +\ldots\pm\frac{m}{n}\left(\frac{m}{n}-1\right)\left(\frac{m}{n}-2\right)\ldots\left(\frac{m}{n}-\rho\right)\frac{1}{(flx)^{\rho+1}} \end{array}\right].$$

$$-\frac{T}{f^{\frac{m}{n}+1}} : \frac{\frac{m}{n}\pi}{\text{sin.}\,\frac{m}{n}\pi} \qquad +$$

$$+\frac{R}{f^{\frac{m}{n}+1}}\left[\begin{array}{l} 1+\frac{m}{n+m}\cdot\frac{flx}{\rho+1}+\frac{m}{2n+m}\cdot\frac{(flx)^2}{(\rho+1)(\rho+2)}+\frac{m}{3n+m}\cdot\frac{(flx)^3}{(\rho+1)(\rho+2)(\rho+3)}+\ldots \\ -\frac{m}{n-m}\cdot\frac{\rho}{2n-m}-\frac{m}{2n-m}\cdot\frac{\rho(\rho-1)}{(flx)^2}-\frac{m}{3n-m}\cdot\frac{\rho(\rho-1)(\rho-2)}{(flx)^3}-\ldots \end{array}\right]$$

Cum ergo alterutra ex duabus aequationibus (10), & (11) exhibeat series convergentes: habebimus pro quocumque valore x non excluso valorem formulae integralis $\int x^{f-1}dx(lx)^{\frac{m}{n}}$, in qua n est numerus impar; sumpta constante ita ut integrale annihiletur posito $x = 0$.

Sumatur nunc in eadem formula prima pro n numerus par; quo casu excluduntur omnes valores ipsius x unitate minores, ac proinde etiam $x = 0$. Praestabit ergo in hoc casu sumere constantem ea lege ut integrale evanescat posito $x = 1$ ob rationem supra allatam.

Si vero fiat $x = 1$; erit $y = flx = 0$; quare si $\nu + 1$, seu $\frac{m}{n} + 1$ sit quantitas positiva; invenietur in aequatione (1) constans $C = 0$; itaque substitutis valoribus $y = flx$; $\nu = \frac{m}{n}$; ob $\int x^{f-1}dx(lx)^{\frac{m}{n}} = \frac{1}{f^{\frac{m}{n}+1}}\int e^{y}y^{\nu}dy$ habebitur pro casu quo n sit numerus par; $\frac{m}{n} + 1$ sit quantitas positiva, atque integrale debeat annihilari quando $x = 1$

$$(12) \ldots\ \int x^{f-1}dx(lx)^{\frac{m}{n}} =$$

$$nlx\left(\frac{1}{n+m}+\frac{1}{2n+m}flx+\frac{1}{2}\cdot\frac{1}{3n+m}(flx)^2+\frac{1}{2.3}\cdot\frac{1}{4n+m}(flx)^3+\ldots\right)$$

Si series (12) non satis convergat ad exhibendum

 valo-

valorem integralis; substituetur aequatio (11) omisso tantum termino $-\frac{T}{f^{\frac{m}{n}+1}} \cdot \frac{\frac{m}{n}\pi}{\text{sin.}\frac{m}{n}\pi}$. Est enim aequatio (11) ejusdem valoris cum (10). Sed (10) dempto termino constanti est ipsa (12); ergo (11) dempto eodem termino erit ejusdem valoris cum (12). Pro casu itaque quod series (12) non satis convergat; adhibebitur

$$(13)\ldots\int x^{f-1}dx(lx)^{\frac{m}{n}} = (11) + \frac{T}{f^{\frac{m}{n}+1}} \cdot \frac{\frac{m}{n}\pi}{\text{sin.}\frac{m}{n}\pi}$$

Quibus omnibus absoluta est evolutio primae formulae. Transeamus nunc, ad alteram formulam

$$\int x^{f-1}dx\left(l\frac{1}{x}\right)^{\frac{m}{n}} = \int x^{f-1}dx\left(-lx\right)^{\frac{m}{n}}$$

atque in primis si n sit numerus impar; erit

$$\int x^{f-1}dx(-lx)^{\frac{m}{n}} = -\int x^{f-1}dx\,(lx)^{\frac{m}{n}}$$

quare valor hujus formulae integralis idem erit mutato tantum signo cum valore primae formulae, in qua item sit n numerus impar.

Sit nunc in secunda formula n numerus par; ac proinde excludantur valores omnes ipsius x praeter positivos, qui continentur intra limites 0, & 1.

Fiat $-lx = \frac{z}{f}$; $\frac{m}{n} = \nu$; erit $lx = \frac{-z}{f}$; $x = c^{-\frac{z}{f}}$

$\int x^{f-1}dx\,(-lx)^{\frac{m}{n}} = -\frac{1}{f^{\nu+1}}\int c^{-z}z^{\nu}dz$. Est vero

$$\int c^{-z}z^{\nu}dz = \int z^{\nu}dz\left(1 - z + \frac{z^2}{2} - \frac{z^3}{2\cdot 3} + \ldots\ldots\right)$$

ac

ac proinde

$$(14)\ldots\int e^{-z} z^{\nu} dz = M + \frac{1}{\nu+1} z^{\nu+1} - \frac{1}{\nu+2} z^{\nu+2} + \frac{1}{\nu+3} \cdot \frac{z^{\nu+3}}{2} + \frac{1}{\nu+4} \cdot \frac{z^{\nu+4}}{2.3} - \ldots\ldots$$

Rursus

$$(15)\ldots\int e^{-z} z^{\nu} dz = -z^{\nu} e^{-z} + \int \nu z^{\nu-1} e^{-z} dz$$

$$(16) \qquad = -z^{\nu} e^{-z} - \nu z^{\nu-1} e^{-z} + \int \nu(\nu-1) z^{\nu-2} e^{-z} dz$$

$$(17) \qquad = -z^{\nu} e^{-z} - \nu z^{\nu-1} e^{-z} - \nu(\nu-1) z^{\nu-2} e^{-z} + \int \nu(\nu-1)(\nu-2) z^{\nu-3} e^{-z} dz$$

&c.

$$(18)\ldots = -z^{\nu} e^{-z} - \nu z^{\nu-1} e^{-z} - \nu(\nu-1) z^{\nu-2} e^{-z} - \ldots - \nu(\nu-1)(\nu-2)\ldots\ldots(\nu-\mu) z^{\nu-\mu-1} e^{-z} + \int \nu(\nu-1)(\nu-2)\ldots(\nu-\mu-1) z^{\nu-\mu-2} e^{-z} dz$$

Quaerenda nunc est pro aequatione (14) constans M talis, ut integrale evanescat posito $z = \infty$; sive $x = 0$. Si evolvantur per substitutionem

$$e^{-z} = 1 - z + \frac{z^2}{2} - \&c.$$

termini sumatorii, qui sunt in secundo membro aequationum (15), (16), (17), (18); habebitur ex (15)

$$(19)\ldots\int e^{-z} z^{\nu} dz = -z^{\nu} e^{-z} + N + z^{\nu} - \frac{\nu}{\nu+1} z^{\nu+1} + \frac{\nu}{\nu+2} \cdot \frac{z^{\nu+2}}{2} - \ldots$$

ex (16) habebitur

$$(20)\ldots\int e^{-z} z^{\nu} dz = -z^{\nu} e^{-z} - \nu z^{\nu-1} e^{-z} + P + \nu z^{\nu-1} - (\nu-1) z^{\nu} + \frac{\nu(\nu-1)}{\nu+1} \cdot \frac{z^{\nu+1}}{2} - \ldots\ldots$$

ex (17) habebitur

(21)

$$(21)\ldots\int e^{-z} z^{\nu} dz = -z^{\nu} e^{-z} - \nu z^{\nu-1} e^{-z} - \nu(\nu-1) z^{\nu-2} e^{-z} + Q$$

$$+ \nu(\nu-1) z^{\nu-2} - \nu(\nu-2) z^{\nu-1} + (\nu-1)(\nu-2)\frac{z^{\nu}}{2}$$

$$- \ldots\ldots$$

tandem ex (18) habebitur

$$(22)\ldots\int e^{-z} z^{\nu} dz = -z^{\nu} e^{-z} - \nu z^{\nu-1} e^{-z} - \nu(\nu-1) z^{\nu-2} e^{-z} - \ldots$$

$$- \nu(\nu-1)(\nu-2)\ldots(\nu-\mu)\, z^{\nu-\mu-1} e^{-z} + S$$

$$+ \nu(\nu-1)(\nu-2)\ldots(\nu-\mu-1)\Big[\frac{1}{\nu-\mu-1} z^{\nu-\mu-1}$$

$$- \frac{1}{\nu-\mu} z^{\nu-\mu} + \frac{1}{\nu-\mu+1} \cdot \frac{z^{\nu-\mu+1}}{2} - \ldots$$

$$\pm \frac{1}{\nu-\mu+\rho-1} \cdot \frac{z^{\nu-\mu+\rho-1}}{2\cdot3\cdot4\ldots\rho} \mp \ldots\ldots\Big]$$

Ubi si constantes N, P, Q, S ingressae per integrationem sumantur ea lege, ut integrale evanescat posito $x = 0$, qua lege sumpta est etiam constans M in aequatione (14); invenietur per methodum adhibitam pro constantibus C, D, E, & K in aequationibus (1), (6), (7), & (8) esse $M = N = P = Q = S$.

Jam ergo inquiramus constantem S. Cum sumpto $z = \infty$ terminus $- z^{\nu} e^{-z}$ in aequatione (22) fit infinite parvus; reliqui vero positi ante constantem S convergant donec $\frac{\nu-\mu}{z}$, qui est factor generalis termini sequentis fit fractio; si non sumatur $\mu > z$, tota series posita ante constantem S evanescit. Eo itaque casu erit

$$(23)\ldots S = -\nu(\nu-1)(\nu-2)(\nu-3)\ldots(\nu-\mu-1)\Big[\frac{1}{\nu-\mu-1} z^{\nu-\mu-1}$$

$$-$$

$$-\frac{1}{\nu-\mu}z^{\nu-\mu} + \frac{1}{\nu-\mu+1}\cdot\frac{z^{\nu-\mu+1}}{2} - \dots$$

$$\pm \frac{1}{\nu-\mu+\rho-1}\cdot\frac{z^{\nu-\mu+\rho-1}}{2\cdot3\cdot4\cdots\rho} \mp \dots\dots]$$

in qua ſerie terminus generalis eſt

$$\pm\frac{\nu(\nu-1)(\nu-2)\dots(\nu-\mu-1)}{2\cdot3\cdot4\dots\rho}\cdot\frac{z^{\nu-\mu+\rho-1}}{\nu-\mu+\rho-1}$$

Sit $\mu+1=\rho=z$, ſitque ρ numerus par; terminus generalis abibit in ſequentem

$\frac{(\nu-1)}{1}\cdot\frac{(\nu-2)}{2}\cdot\frac{(\nu-3)}{3}\dots\frac{(\nu-\rho)}{\rho}\rho^{\nu}$, qui fiat $=T$. Series ſequens erit

$$-\frac{\nu}{\nu+1}\cdot\frac{z}{\rho+1}T+\frac{\nu}{\nu+2}\cdot\frac{z^2}{(\rho+1)(\rho+2)}T-$$
$$\frac{\nu}{\nu+3}\cdot\frac{z^3}{(\rho+1)(\rho+2)(\rho+3)}T+\dots$$

Antecedens vero retrocedendo ab ipſo termino T erit

$$-\frac{\nu}{\nu-1}\cdot\frac{\rho}{z}T+\frac{\nu}{\nu-2}\frac{\rho(\rho-1)}{z^2}T-\frac{\nu}{\nu-3}\cdot\frac{\rho(\rho-1)(\rho-2)}{z^3}T+\dots$$

Quare cum ſit ratione infiniti $\frac{z}{\rho+1}=1=$
$\frac{z^2}{(\rho+1)(\rho+2)}=\frac{z^3}{(\rho+1)(\rho+2)(\rho+3)}$ &c. $=\frac{\rho}{z}=\frac{\rho(\rho-1)}{z^2}=$
$\frac{\rho(\rho-1)(\rho-2)}{z^3}$ &c. erit tandem

$$S=-T\left[\begin{array}{l}1-\frac{\nu}{\nu+1}+\frac{\nu}{\nu+2}-\frac{\nu}{\nu+3}+\dots\\ -\frac{\nu}{\nu-1}+\frac{\nu}{\nu-2}-\frac{\nu}{\nu-3}+\dots\end{array}\right]$$

ſeu

seu $S = - T \frac{\nu\pi}{\text{sin.}\,\nu\pi}$ (Vide superius posita ante aequationem (10)).

Cum ergo sit $M = S$; substitutis valoribus $z = -flx$ $\nu = \frac{m}{n}$ in aequatione (14); habebimus ob

$$\int x^{f-1} dx\,(-lx)^{\frac{m}{n}} = -\frac{1}{f^{\frac{m}{n}+1}} \int e^{-z} z^{\nu} dz$$

$$(24) \ldots \int x^{f-1} dx\,(-lx)^{\frac{m}{n}} = \frac{T}{f^{\frac{m}{n}+1}} \cdot \frac{\frac{m}{n}\pi}{\text{sin.}\,\frac{m}{n}\pi}$$

$$-n(-lx)^{\frac{m}{n}+1} \left[\begin{array}{l} \frac{1}{n+m} + \frac{1}{2n+m} flx + \frac{1}{3n+m} \cdot \frac{1}{2} (flx)^2 \\ + \frac{1}{4n+m} \cdot \frac{1}{2.3} (flx)^3 + \frac{1}{5n+m} \cdot \frac{1}{2.3.4} (flx)^4 + \ldots \end{array} \right]$$

ubi est $T = \frac{n-m}{n} \cdot \frac{2n-m}{2n} \cdot \frac{3n-m}{3n} \ldots \frac{\rho n-m}{\rho n} \rho^{\frac{m}{n}}$

facto $\rho = \infty$.

Series (24) convergit quoties flx est quantitas fracta; si vero non satis convergat; adhibenda erit series (22) praeparata ut sequitur.

Sumpto $\mu + 1 = \rho$ numero integro positivo proximiori valori ipsius z; terminus generalis seriei positae post constantem S in aequatione (22) fiet, ut vidimus

$\pm \frac{(\nu-1)}{1} \cdot \frac{(\nu-2)}{2} \cdot \frac{(\nu-3)}{3} \ldots \frac{(\nu-\rho)}{\rho} z^{\nu}$, qui fiat $= \pm V$;

erit series sequens ad dexteram

$$\mp \frac{\nu}{\nu+1} \cdot \frac{z}{\rho+1} V \pm \frac{\nu}{\nu+2} \cdot \frac{z^2}{(\rho+1)(\rho+2)} V \mp \frac{\nu}{\nu+3} \cdot \frac{z^3}{(\rho+1)(\rho+2)(\rho+3)} V \pm \ldots$$

ad sinistram vero retrocedendo erit

$\mp$

$$\mp \frac{\nu}{\nu-1} \cdot \frac{\rho}{z} V \pm \frac{\nu}{\nu-2} \cdot \frac{\rho(\rho-1)}{z^2} V \mp \frac{\nu}{\nu-3} \cdot \frac{\rho(\rho-1)(\rho-2)}{z^3} V \pm \ldots$$

quae ſeries ambae convergunt; ſecunda vero etiam eſt finita.

Sumpto ergo $V = \frac{m-n}{n} \cdot \frac{m-2n}{2n} \cdot \frac{m-3n}{3n} \ldots \frac{m-\rho n}{\rho n} (-f\,lx)^{\frac{m}{n}}$

$$\text{ob} \int x^{f-1} dx\, (-lx)^{\frac{m}{n}} = -\frac{1}{f^{\nu+1}} \int c^{-z} z^{\nu} dz$$

erit ex aequatione (22)

$$(25) \ldots \int x^{f-1} dx\, (-lx)^{\frac{m}{n}} = \frac{T}{f^{\frac{m}{n}+1}} \cdot \frac{\frac{m}{n}\pi}{\text{ſin.}\, \frac{m}{n}\pi}$$

$$+ \frac{(-lx)^{\frac{m}{n}} x^f}{f} \left[\begin{array}{l} 1 + \frac{m}{n} \frac{1}{(-f\,lx)} + \frac{m}{n}\left(\frac{m}{n}-1\right) \frac{1}{(-f\,lx)^2} + \ldots \\ + \frac{m}{n}\left(\frac{m}{n}-1\right)\left(\frac{m}{n}-2\right) \ldots \left(\frac{m}{n}-\mu\right) \frac{1}{(-f\,lx)^{\mu+1}} \end{array} \right]$$

$$\pm m V \left[\begin{array}{l} \frac{1}{m} + \frac{1}{n+m} \cdot \frac{f\,lx}{\rho+1} + \frac{1}{2n+m} \cdot \frac{(f\,lx)^2}{(\rho+1)(\rho+2)} + \ldots \\ - \frac{1}{n-m} \frac{\rho}{f\,lx} - \frac{1}{2n-m} \cdot \frac{\rho(\rho-1)}{(f\,lx)^2} - \ldots \end{array} \right]$$

Cum ergo alterutra ex duabus aequationibus (24), & (25) exhibeat ſeries convergentes; habebimus pro quocumque valore x inter 0, & 1 (nam caeteri valores excluduntur) valorem formulae integralis

$$\int x^{f-1} dx\, (-lx)^{\frac{m}{n}}$$

in qua n eſt numerus par; ſumpta conſtante ita ut annihiletur integrale poſito $x = 0$.

Corol.

Corollarium.

Cum fit $T = \frac{n-m}{n} \cdot \frac{2n-m}{2n} \cdot \frac{3n-m}{3n} \cdots \frac{\lambda n-m}{\lambda n} \cdots \frac{\rho n-m}{\rho n} \rho^{\frac{m}{n}}$

posito ρ numero infinito; si fit $n = 2$; fit vero numerus impar $m = 2\lambda - 1$; erit

$$T = \frac{2-m}{2} \cdot \frac{4-m}{4} \cdot \frac{6-m}{6} \cdots \frac{2\lambda-m}{2\lambda} \cdots \frac{2\rho-m}{2\rho} \rho^{\frac{m}{2}} =$$

$$(2-m)(4-m(6-m) \cdots (2(\lambda-1)-m) \cdot \frac{2\lambda-m}{2} \cdot \frac{2(\lambda+1)-m}{4} \cdots \times$$

$$\frac{2\rho-m}{2\rho-m+1} \rho^{\frac{1}{2}} \cdot \frac{\rho^{\frac{m-1}{2}}}{(2\rho-m+3)(2\rho-m+5) \cdots (2\rho-m+2(\lambda-1)+1)}$$

$$= (3-2\lambda) \cdots (-9)(-7)(-5)(-3)(-1) \cdot \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdots \times$$

$$\frac{2\rho-m}{2\rho-m+1} \rho^{\frac{1}{2}} \cdot \frac{\rho^{\lambda-1}}{(2\rho)^{\lambda-1}} =$$

$$\frac{3-2\lambda}{2} \cdots \frac{(-9)}{2} \cdot \frac{(-7)}{2} \cdot \frac{(-5)}{2} \cdot \frac{(-3)}{2} \cdot \frac{(-1)}{2} \cdot \sqrt{\frac{1}{\pi}} \text{ ob}$$

$$\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{7}{8} \cdots \frac{2\rho-m}{2\rho-m+1} \rho^{\frac{1}{2}} = \sqrt{\frac{1}{\pi}}.$$

Erit quoque $\frac{T}{f^{\frac{m}{n}+1}} \cdot \frac{\frac{m}{n}\pi}{\text{fin.} \frac{m}{n}\pi} = \pm T \frac{\left(\lambda - \frac{1}{2}\right)\pi}{f^{\lambda+\frac{1}{2}}}$

ubi sumendum est signum superius $+$ si λ fuerit impar; inferius vero $-$; si λ fuerit par.

Cum vero valor ipsius $T = \frac{3-2\lambda}{2} \cdots \frac{(-5)}{2} \cdot \frac{(-3)}{2} \cdot \frac{(-1)}{2} \sqrt{\frac{1}{\pi}}$

fit

sit positivus si λ fuerit impar; negativus, si par; erit

$$\frac{T}{f^{\frac{m}{n}+1}} \cdot \frac{\frac{m}{2}\pi}{\text{sin.}\,\frac{m}{2}\pi}$$ semper quantitas positiva

$$= \frac{2\lambda-3}{2} \,\ldots\ldots\, \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \sqrt{\frac{1}{\pi}} \times \frac{\lambda-\frac{1}{2}}{f^{\lambda+\frac{1}{2}}}\pi =$$

$$\frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{2} \cdot \frac{7}{2} \ldots\ldots \frac{2\lambda-1}{2} \cdot \frac{\sqrt{\pi}}{f^{\lambda+\frac{1}{2}}}$$

Quare facto post integrationem $x = 1$; habebitur ex aequatione (24)

$$\int x^{f-1} dx \left(l\,\frac{1}{x}\right)^{\frac{2\lambda-1}{2}} = \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{2} \ldots\ldots \frac{2\lambda-1}{2} \cdot \frac{\sqrt{\pi}}{f^{\lambda+\frac{1}{2}}}$$

quod consentaneum est inventis Euleri (In Comment. cit. §. 29.).

Evolutio formularum

$$\int \frac{dx\,lx}{\sqrt{(1-xx)}} \quad , \quad \int \frac{dx\,lx}{1+xx}$$

Quod attinet ad primam formulam, in Actis Acad. Scient. Petrop. ad annum 1777. Tom. I. Part. II. pag. 3. Eulerus inseruit Commentarium hujus tituli : *de integratione formulae* $\int \frac{dx\,lx}{\sqrt{(1-x^2)}}$ *ab* $x = 0$, *ad* $x = 1$ *extensa*. De hac autem integratione haec habet : *cum nuper singulari methodo invenissem esse* $\int \frac{dx\,lx}{\sqrt{(1-xx)}} \left(\begin{array}{l} ab\ x = 0 \\ ad\ x = 1 \end{array}\right) = -\frac{1}{2}\pi\, l\, 2$ *ex-*

expressio integralis eo majori attentione digna est censenda, quod ejus investigatio neutiquam est obvia; unde operae pretium esse duxi ejus veritatem etiam ex aliis fontibus ostendisse, antequam ipsam methodum, quae me eo perduxit exponerem. Propositae vero integrationis affert demonstrationes tres. Nos omissa prima, quae exhibet valorem integralis dumtaxat pro casu $x = 1$, alias duas illustrabimus. Methodus, quam deinceps explicat, ad alia praeclara inventa perducit, quae non sunt hujus loci.

In primis vero notandi sunt limites, quibus continetur valor realis formulae differentialis propositae, qui iidem observandi sunt etiam in integrali; pro iis enim valoribus x, pro quibus formula differentialis fit imaginaria; censendum est etiam integrale fieri imaginarium, ut fluxio sit quantitas analoga quantitati fluenti. Itaque ratione lx, qui ingreditur formulam differentialem, x non poterit esse negativa; ratione vero $\sqrt{(1-xx)}$, x non potest superare unitatem. Quare integrale ingredi non poterunt nisi valores x positi intra limites 0, & 1. Nunc secunda demonstratio Euleri est hujusmodi.

Facto $x = \text{sin.}\, \varphi$; cum sit $\frac{dx}{\sqrt{(1-xx)}} = d\varphi$; habetur

$$\int \frac{dx\, lx}{\sqrt{(1-xx)}} = d\varphi\, l.\, \text{sin.}\, \varphi$$

Est autem

$$l.\,\text{sin.}\,\varphi = -l2 - \text{cos.}\, 2\varphi - \frac{1}{2}\, \text{cos.}\, 4\varphi - \frac{1}{3}\, \text{cos.}\, 6\varphi - \&c.$$

(Calcul. Integr. Vol. I. §. 296.); erit ergo

$$(E) \ldots \int d\varphi\, l.\, \text{sin.}\, \varphi = -\varphi\, l2 - \frac{2\,\text{sin.}\,2\varphi}{2^2} - \frac{2\,\text{sin.}\,4\varphi}{4^2} - \frac{2\,\text{sin.}\,6\varphi}{6^2} - \ldots$$

quae expressio evanescit posito $x = \text{sin.}\, \varphi = 0$

Quod si capiatur $\varphi = 90^\circ = \frac{\pi}{2}$; seu $x = 1$; habe-

habetur $\int \frac{dx\, lx}{\sqrt{(1-xx)}} \left(\begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = 1 \end{array} \right) = -\frac{1}{2} \pi l 2$

Subdit vero Eulerus. *Ista autem demonstratio praecedenti ideo longe antecellit, quod nobis non solum valorem formulae propositae exhibeat casu quo* $\phi = 90°$, *sed etiam verum ejus valorem ostendat, quicumque angulus pro* ϕ *accipiatur, id quod ad ipsam formulam propositam* $\int \frac{dx\, lx}{\sqrt{(1-xx)}}$ *transferri potest; cujus adeo valorem, pro quolibet valore ipsius* x *assignare poterimus. Quod si enim istius formulae valorem desideremus ab* $x = 0$ *usque ad* $x = a$; *quaeratur angulus* α, *cujus sinus sit aequalis ipsi* a, *atque semper habebitur*

$$\int \frac{dx\, lx}{\sqrt{(1-xx)}} \left(\begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = a \end{array} \right) = -\alpha l 2 - \frac{2\text{fin.}2\alpha}{2^2} - \frac{2\text{fin.}4\alpha}{4^2} - \frac{2\text{fin.}6\alpha}{6^2} - \text{etc.}$$

Unde patet quoties fuerit $\alpha = \frac{i\pi}{2}$ *denotante* i *numerum integrum quemcumque; quoniam omnes sinus evanescunt, valorem formulae his casibus finite exprimi per* $-\frac{i\pi}{2} l 2$; *aliis vero casibus valor nostrae formulae per seriem infinitam satis concinnam exprimetur.*

Cui doctrinae haec opponi possunt: quoties fuerit $\alpha = \frac{i\pi}{2}$, erit $x = a = \text{fin.}\,\alpha = \text{fin.}\,\frac{i\pi}{2}$ aequalis aut ipsi o, aut $= +1$, aut $= -1$. Nempe

Si i fuerit numerus par $= 2m$; erit $a = 0$; ac proinde

$$\int \frac{dx\, lx}{\sqrt{(1-xx)}} \left(\begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = 0 \end{array} \right) = -m\pi l 2$$

ubi m indeterminate significat numeros omnes integros positivos, aut negativos.

Si i fuerit numerus formae $4m+1$; erit $a = 1$; ac pro-

proinde $\int \frac{dx\,lx}{\sqrt{(1-xx)}} \left(\begin{matrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{matrix} \right) = -\frac{(4m+1)}{2} \pi\, l\, 2$

Si demum fuerit numerus i formae $4m - 1$; erit $a = -1$; ac propterea

$$\int \frac{dx\,lx}{\sqrt{(1-xx)}} \left(\begin{matrix} \text{ab } x = 0 \\ \text{ad } x = -1 \end{matrix} \right) = -\frac{(4m-1)}{2} \pi\, l\, 2.$$

Hae autem constantes ita vagae videntur absurdae pluribus nominibus; secunda vero $-\frac{(4m+1)}{2} \pi\, l\, 2$, quae exhibet valorem integralis ab $x = 0$ ad $x = 1$ confundit cum infinitis aliis integrale ab Eulero supra assignatum.

Ut haec explicentur, notandum est in serie aequationis (E) non omnes arcus pro φ assumi posse. Cum enim incipiamus integrationem a $\varphi = 0$, licebit sumere dumtaxat arcus, qui sunt inter $\varphi = 0$, & $\varphi = \pi$; tunc enim $d\varphi\, l.\text{sin.}\varphi$ perpetuo habet valorem realem. Quod si arcus φ ulterius fluere cogeretur; tunc ejus sinus fieret negativus, ac proinde ex recepta doctrina ipsius Euleri $d\varphi\, l.\text{sin.}\varphi$ fieret imaginarium. Ergo integratio realis formulae $\int d\varphi\, l.\text{sin.}\varphi$ includitur intra terminos $\varphi = 0$, & $\varphi = \pi$, neque ultra porrigitur; habetur vero

$$\int d\varphi\, l.\text{sin.}\varphi \left(\begin{matrix} \text{a } \varphi = 0 \\ \text{ad } \varphi = \pi \end{matrix} \right) = -\pi\, l\, 2$$

Restat explicandum quomodo substitutis valoribus in hac ultima aequatione, intelligenda sit aequatio, quae oritur

$$\int \frac{dx\,lx}{\sqrt{(1-xx)}} \left(\begin{matrix} \text{ab } x = 0 \\ \text{ad } x = 0 \end{matrix} \right) = -\pi\, l\, 2$$

Sed cum uti vidimus in hac formula valor x dumtaxat a 0 usque ad 1, possit excrescere; huic autem fluxui respondeat fluxus φ a 0 usque ad $\frac{\pi}{2}$; in aequatione (E), quae subsidiaria dici potest, valor accipiendus pro φ inclusus cense-

censebitur intra limites $\varphi = 0$, & $\varphi = \frac{\pi}{2}$; quare excludetur aequatio incongrua

$$\int \frac{dx\,lx}{\sqrt{(1-xx)}} \left(\begin{matrix}\text{ab } x = 0 \\ \text{ad } x = 0\end{matrix}\right) = -\pi l 2$$

ad quam habendam fluere fecimus φ ultra valorem $\frac{\pi}{2}$ usque ad valorem π. Haec tamen omnia submitto sapientiorum judicio

Intra limites vero $\varphi = 0$, & $\varphi = \frac{\pi}{2}$ aequatio (E) mirifice inservit ad habendum valorem integralis $\int \frac{dx\,lx}{\sqrt{(1-xx)}}$ pro quocumque valore x, quem contineri vidimus inter 0, & 1. Series enim aequationis (E) semper convergit. Ad hujus seriei *egregiae* analogiam, quae Eulero *se quasi praeter expectationem obtulit*, nos aliam formulam evolvemus.

Tertia demonstratio Euleri ita se habet. Fiat $x = \text{cos.}\,\varphi$; erit $\int \frac{dx\,lx}{\sqrt{(1-xx)}} = -\int d\varphi\, l.\text{cos.}\,\varphi$, *quod integrale a* $\varphi = 90°$, *usque ad* $\varphi = 0$ *erit extendendum*. Est autem $l.\text{cos.}\,\varphi = -l2 + \text{cos.}\,2\varphi - \frac{1}{2}\text{cos.}\,4\varphi + \frac{1}{3}\text{cos.}\,6\varphi - \frac{1}{4}\text{cos.}\,8\varphi + \ldots$ (Euler. Calc. Integr. Vol. I. §. *296.*). Unde fit

(F) ... $\int d\varphi\, l.\text{cos.}\,\varphi = C + \varphi l 2 - \frac{2\,\text{sin.}\,2\varphi}{2^2} + \frac{2.\text{sin.}\,4\varphi}{4^2} - \frac{2\,\text{sin.}\,6\varphi}{6^2} + \ldots$

Cum integrale evanescere debeat casu $x = 0$, seu $\varphi = 90°$; invenietur $C = -\frac{\pi}{2} l 2$. In serie autem (F) φ est complementum illius anguli, qui designabatur per φ in aequatione (E). Quare eosdem limites habebit ordine inverso.

Transea-

Transeamus nunc ad secundam formulam; Cum sit $\frac{dx}{1+xx} = d.$ A tang. x; si fiat A tang. $x = \phi$; erit $\int \frac{dx\, lx}{1+xx} = \int d\phi\, l.$ tang. ϕ. Est autem (Euler. Calcul. Integr. Vol. I. §. 296.) $l.$ tang. $\phi = -2$ cos. $2\phi - \frac{2}{3}$ cos. $6\phi - \frac{2}{5}$ cos. $10\phi - \frac{2}{7}$ cos. $14\phi - \ldots\ldots$ Erit ergo

(H) ... $\int d\phi\, l.\text{tang.}\phi = -4\left(\frac{\text{sin.}2\phi}{2^2} + \frac{\text{sin.}6\phi}{6^2} + \frac{\text{sin.}10\phi}{10^2} + \frac{\text{sin.}14\phi}{14^2} + \ldots\right)$

sine additione constantis cum series annihiletur posito $x = \text{tang.}\ \phi = 0$.

Si capiatur $x = 1$, seu $\phi = \frac{\pi}{4}$; erit

$$\int d\phi\, l.\ \text{tang.}\ \phi = -1 + \frac{1}{3^2} - \frac{1}{5^2} + \frac{1}{7^2} - \frac{1}{9^2} + \ldots$$

Si capiatur $x = \infty$ seu $\phi = \frac{\pi}{2}$; erit rursus $\int d\phi\, l.$ tang. $\phi = 0$. Id autem facile explicatur. Differentiale enim $d\phi\, l.$ tang. ϕ manente positivo $d\phi$ in perpetuis augmentis arcus ϕ, ratione $l.$ tang. ϕ est negativum a valore $\phi = 0$, usque ad $\phi = \frac{\pi}{4}$ ubi differentiale annihilatur; quare ejus integrale ab $x = 0$ ad $x = 1$ est negativum. Cum vero deinde a valore $\phi = \frac{\pi}{4}$ usque ad $\phi = \frac{\pi}{2}$, valor $l.$ tang. ϕ sit positivus, etiam fluxio sit positiva, ac proinde destruit effectum negativae anterioris in integrali, donec in limite $\phi = \frac{\pi}{2}$ integrale annihiletur.

Non

Non possumus autem in aequatione (H) sumere arcum $\varphi > \frac{\pi}{2}$, ne in formulam differentialem ingrediatur logarithmus quantitatis negativae.

Cum sumpto post integrationem $\varphi = \frac{\pi}{4}$ sit
$4\int\varphi d\varphi = \frac{\pi\pi}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} +$ &c. (Euler. Introd. in Anal. §. 156.); erit
$\frac{1}{3^2} + \frac{1}{7^2} + \frac{1}{11^2} + \frac{1}{15^2} +$ &c. $= \int d\varphi\left(2\varphi + \frac{1}{2} l.\text{tang}.\varphi\right)$
integratione a $\varphi = 0$ ad $\varphi = \frac{\pi}{4}$ extensa. Atque item
$1 + \frac{1}{5^2} + \frac{1}{9^2} + \frac{1}{13^2} +$ &c. $\int d\varphi\left(2\varphi - \frac{1}{2} l.\text{tang}.\varphi\right)$.

Evolutio formulae

$$\int \frac{d\varphi}{\text{l. tang}.\varphi}$$

Fiat tang. $\varphi = x$; erit $\int \frac{d\varphi}{l.\text{ tang}.\varphi} = \int \frac{dx}{(1+xx)lx}$. Ut hujusmodi integratio absolvatur notetur prius facto $x^{m+1} = u$ esse $\int \frac{x^m dx}{lx} = \int \frac{du}{lu}$. Cum vero sit ex Adnotatione I.

$\int \frac{du}{lu} = A + l \pm lu + lu + \frac{(lu)^2}{2.2} + \frac{(lu)^3}{2.3.3} + \frac{(lu)^4}{2.3.4.4} + \ldots\ldots$ posito $A = 0,577215$, &c. ut integrale evanescat casu $u = x^{m+1} = 0$ seu $x = \text{tang}.\varphi = 0$, seu tandem $\varphi = 0$; erit

erit $\int \frac{dx}{(1+xx)lx} = \int \frac{dx}{lx}(1-x^2+x^4-x^6+x^8-x^{10}+\ldots)$

$= A + \quad l\pm lx + lx + \frac{(lx)^2}{2.2} + \frac{(lx)^3}{2.3.3} + \frac{(lx)^4}{2.3.4.4} + \ldots\ldots$

$- A - l3 - l\pm lx - 3lx - \frac{3^2(lx)^2}{2.2} - \frac{3^3(lx)^3}{2.3.3} - \frac{3^4(lx)^4}{2.3.4.4} - \ldots\ldots$

$+ A + l5 + l\pm lx + 5lx + \frac{5^2(lx)^2}{2.2} + \frac{5^3(lx)^3}{2.3.3} + \frac{5^4(lx)^4}{2.3.4.4} + \ldots\ldots$

$- A - l7 - l\pm lx - 7lx - \frac{7^2(lx)^2}{2.2} - \frac{7^3(lx)^3}{2.3.3} - \frac{7^4(lx)^4}{2.3.4.4} - \ldots\ldots$

$+$ &c.

Est autem $A - A + A - A +$ &c. in infin. $= \frac{1}{2} A$

$-l3+l5-l7+l9-\ldots = l\frac{1.5.9.13\ldots\rho}{3.7.11.15\ldots(\rho+2)}\sqrt{(\rho+4)}$

facto $\rho = \infty$; ad quem valorem proximius accedimus quo major sumitur inter finitos numerus $\rho = 4n-3$ existente n indice factorum. Nam sumpto $n = \infty$ est

$-l3+l5-l7+l9-\ldots+l(4n-3)-l(4n-1)+l(4n+1)-l(4n+3)+\ldots$

$= -l3+l5-l7+l9-\ldots+l(4n-3)-l(4n-1)+\frac{1}{2}l(4n+1)$

ob $l(4n+3) = l(4n+5) = l(4n+7) =$ &c. $= l(4n+1)$

Est item $l\pm lx - l\pm lx + l\pm lx -$ &c. $= \frac{1}{2}\, l\pm lx$

$1-3+5-7+9-$ &c.... $= 0$

$1-3^2+5^2-7^2+9^2-$ &c. $= -\frac{1}{2}$

$1-3^3+5^3-7^3+9^3-$ &c. $= 0$

$1-3^4+5^4-7^4+9^4-$ &c. $= \frac{5}{2}$

$1-3^5+5^5-7^5+9^5-$ &c. $= 0$

$1-3^6+5^6-7^6+9^6-$ &c. $= -\frac{61}{2}$

$1-3^7+5^7-7^7+9^7-$ &c. $= 0$

&c.

Nam

Nam cum ſit

$$(1)\quad \frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - \&c.$$

facto $x = 1$ erit $\frac{1}{2} = 1 - 1 + 1 - 1 + 1 - \&c.$

Multiplicetur aequatio (1) per x; differentietur, ac dividatur per dx; reſultabit

$$(2)\quad \frac{1-x^2}{(1+x^2)^2} = 1 - 3x^2 + 5x^4 - 7x^6 + 9x^8 - \&c.$$

ubi facto $x = 1$ erit $0 = 1 - 3 + 5 - 7 + 9 - \&c.$

Multiplicetur aequatio (2) per x; differentietur ac dividatur per dx; reſultabit

$$(3)\quad \frac{1-6x^2+x^4}{(1+x^2)^3} = 1 - 3^2x^2 + 5^2x^4 - 7^2x^6 + 9^2x^8 - \&c.$$

ubi facto $x = 1$ erit $-\frac{1}{2} = 1 - 3^2 + 5^2 - 7^2 + 9^2 - \ldots$

Atque eadem methodo deinceps invenientur alii valores ſequentium ſerierum.

Brevius tamen iidem valores reperientur per regulam ſequentem erutam ex lege calculi ſuperioris. Multiplicentur ſinguli coefficientes numeratoris unius fractionis per ſingulos terminos ſeriei 1 3 5 7 9 &c. per ordinem directum; tum iidem per ordinem inverſum incipiendo ab ultimo termino ſeriei 1 3 5 7 9 &c. prius adhibito. Secunda ſeries productorum ſubtrahatur a prima ponendo primum terminum ſecundae ſeriei ſub ſecundo primae, ſecundum ſub tertio &c. Habebitur nova ſeries coefficientium pro numeratore fractionis ſequentis. Denominator autem erit productum denominatoris antecedentis ducti in $1 + x^2$.

Exempli cauſa coefficientes in aequatione (2) ſunt 1 & -1. Scribatur $1 - 3$

$\qquad\qquad + 3 - 1$

Ac ſubtracta ſecunda ſerie a prima habentur coefficientes aequationis (3) $1 - 6 + 1$. Denominator autem eſt $(1+x^2)^2$

$(1+x^2)^1 \times (1+x^2) = (1+x^2)^3$ qui ſumpto $x=1$ fit 2^3. Unde habetur valor ſeriei $\frac{1-6+1}{2^3} = -\frac{1}{2}$. Rurſus ad habendos coefficientes pro aequatione ſequenti ſcribatur $1-3.6+5$
$5-3.6+1$
ac facta ſubtractione habebuntur coefficientes $1-23+23-1=0$. Denominator vero erit 2^4. Pro ſequenti aequatione ſcribendo

$$\begin{array}{l} 1-3.23+5.23-7 \\ +7 \quad -5.23+3.23-1 \end{array}$$

habebuntur coefficientes

$$1-76+230-76+1=80$$

qui valor diviſus per denominatorem 2^5 dat pro valore ſeriei $\frac{5}{2}$. Atque ita deinceps.

Rurſus valor $l\,\frac{1.5.9.13\ldots\ldots\rho}{3.7.11.15\ldots(\rho+2)}\sqrt{(\rho+4)}$ ſumpto ρ numero infinito medius eſt inter duos valores $l\,\frac{1.5.9.13\ldots\ldots\rho}{3.7.11.15\ldots(\rho+2)}\sqrt{(\rho+4)}$, & $l\,\frac{1.5.9.13\ldots\ldots}{3.7.11.15\ldots\sqrt{(\rho+2)}}$ ſumpto ρ numero finito. Reperietur tamen facilius hic valor per methodum ab Eulero traditam (Calculo Different. Part. Poſter. Cap. I. §. 11.) $=-0,391594383$.

Quare erit $\int \frac{d\,\phi}{l.\,\text{tang.}\,\phi} = -0,102986551$

$$+\frac{1}{2}\,l \pm l.\,\text{tang.}\,\phi - \frac{1}{2}\,\frac{(l.\,\text{tang.}\,\phi)^2}{2.2} + \frac{5}{2}\,\frac{(l.\,\text{tang.}\,\phi)^4}{2.3.4.4}$$
$$-\frac{61}{2}\,.\,\frac{(l.\,\text{tang.}\,\phi)^6}{2.3.4.5.6.6}\ \&c.$$

Ubi tamen incertum eſt an coefficientes numerici perpetuo convergant, quod invenire operae pretium foret.

Pro valoribus $l.\,\text{tang.}\,\phi$ unitate majoribus alia integratio erui poteſt ex aequatione (10) Adnotationis I.

Sequentia V. C. G. Fontana edenda miſit.

GREGORII FONTANAE

IN REG. TICIN. ARCHIGYMN. SUBLIMIORIS
ANALYSEOS PROF.

Theoremata quatuor ad Calculum Integralem ſpectantia, quae ex Euleriana formula T. I. Calcul. Integr. §. 261 derivantur, ſed novo, longeque faciliori integrandi artificio demonſtrantur.

Theorema I.

$$\int \frac{dx}{\text{ſin.}\, x + \text{ſin.}\, a} = \frac{1}{\text{coſ.}\, a} \log. \frac{\text{ſin.} \frac{1}{2}(x+a)}{\text{coſ.} \frac{1}{2}(x-a)};$$

Dem. Ex Doctrina functionum circularium conſtat; eſſe $\text{ſin.}x + \text{ſin.}\, a = 2\, \text{ſin.}\, \frac{1}{2}(x+a)\, \text{coſ.}\, \frac{1}{2}(x-a)$, hincque $\frac{1}{\text{ſin.}x+\text{ſin.}a} = \frac{1}{2\, \text{ſin.} \frac{1}{2}(x+a)\, \text{coſ.} \frac{1}{2}(x-a)}$. Fractionem hanc binis praeditam factoribus reſolvo in duas haſce $\frac{A\, \text{coſ.} \frac{1}{2}(x+a)}{\text{ſin.} \frac{1}{2}(x+a)} + \frac{B\, \text{ſin.} \frac{1}{2}(x-a)}{\text{coſ.} \frac{1}{2}(x-a)}$, ſumptis A & B coefficientibus indeterminatis. Reductis porro fractionibus ipſis ad communem denominatorem oritur

$$\frac{A\, \text{coſ.} \frac{1}{2}(x+a)\, \text{coſ.} \frac{1}{2}(x-a) + B\, \text{ſin.} \frac{1}{2}(x-a)\, \text{ſin.} \frac{1}{2}(x+a)}{\text{ſin.} \frac{1}{2}(x+a)\, \text{coſ.} \frac{1}{2}(x-a)}$$

$= \frac{1}{2} \cdot \frac{1}{\text{ſin.} \frac{1}{2}(x+a)\, \text{coſ.} \frac{1}{2}(x-a)}$. Jam animadverto, nu-

mera-

meratorem primi membri hujus aequationis posito A = B fieri

$$A\left[\cos.\frac{1}{2}(x+a)\cos.\frac{1}{2}(x-a)+\sin.\frac{1}{2}(x-a)\sin.\frac{1}{2}(x+a)\right]$$

$$=A\cos.\frac{1}{2}\left[(x+a)-(x-a)\right]=A\cos.a,$$ qui, si aequetur numeratori alterius membri $\frac{1}{2}$, praebet $A=B=\frac{1}{2\cos.a}$. Quocirca nancifcimur aequationem

$$\frac{dx}{\sin.x+\sin.a}=\frac{1}{\cos.a}\cdot\frac{\frac{1}{2}dx\cos.\frac{1}{2}(x+a)}{\sin.\frac{1}{2}(x+a)}+\frac{1}{\cos.a}\cdot\frac{\frac{1}{2}dx\sin.\frac{1}{2}(x-a)}{\cos.\frac{1}{2}(x-a)},$$ unde

integrando protinus eruitur $\int\frac{dx}{\sin.x+\sin.a}=\frac{1}{\cos.a}\log.\sin.\frac{1}{2}(x+a)$

$$-\frac{1}{\cos.a}\log.\cos.\frac{1}{2}(x-a)=\frac{1}{\cos.a}\log.\frac{\sin.\frac{1}{2}(x+a)}{\cos.\frac{1}{2}(x-a)}.$$ Q. E. D.

Theorema II.

$$\int\frac{dx}{\sin.x-\sin.a}=\frac{1}{\cos.a}\log.\frac{\sin.\frac{1}{2}(x-a)}{\cos.\frac{1}{2}(x+a)}.$$

Dem. Per nota angulorum theoremata est $\sin.x-\sin.a=2\sin.\frac{1}{2}(x-a)\cos.\frac{1}{2}(x+a)$. Igitur $\frac{1}{\sin.x-\sin.a}=\frac{1}{2\sin.\frac{1}{2}(x-a)\cos.\frac{1}{2}(x+a)}$. Facta vero resolutione in fractiones binas prodit fractio $\frac{1}{\sin.x-\sin.a}=\frac{A\cos.\frac{1}{2}(x-a)}{\sin.\frac{1}{2}(x-a)}+\frac{B\sin.\frac{1}{2}(x+a)}{\cos.\frac{1}{2}(x+a)}$, hisque ad communem denominatorem re-

ductis

ductis nanciscimur aequationem $\frac{1}{2} \cdot \frac{1}{\text{sin.} \frac{1}{2}(x-a) \text{cos.} \frac{1}{2}(x+a)}$
$= \frac{\text{A cos.} \frac{1}{2}(x-a) \text{cos.} \frac{1}{2}(x+a) + \text{B sin.} \frac{1}{2}(x+a) \text{sin.} \frac{1}{2}(x-a)}{\text{sin.} \frac{1}{2}(x-a) \text{cos.} \frac{1}{2}(x+a)}$,
in qua, si capiatur $A = B$, numerator secundi membri abit in $\text{A cos.} \frac{1}{2} \left[(x+a) - (x-a) \right] = \text{A cos.}\, a = \frac{1}{2}$ numeratori prioris membri; proindeque $A = B = \frac{1}{2 \text{cos.}\, a}$.
Quapropter $\frac{dx}{\text{sin.}\, x - \text{sin.}\, a} = \frac{1}{\text{cos.}\, a} \cdot \frac{\frac{1}{2} dx \,\text{cos.} \frac{1}{2}(x-a)}{\text{sin.} \frac{1}{2}(x-a)} +$
$\frac{1}{\text{cos.}\, a} \cdot \frac{\frac{1}{2} dx \,\text{sin.} \frac{1}{2}(x+a)}{\text{cos.} \frac{1}{2}(x+a)}$, & integrando $\int \frac{dx}{\text{sin.}\, x - \text{sin.}\, a} =$
$\frac{1}{\text{cos.}\, a} \text{log. sin.} \frac{1}{2}(x-a) - \frac{1}{\text{cos.}\, a} \text{log.} \frac{1}{2} \text{cos.} (x+a) =$
$\frac{1}{\text{cos.}\, a} \text{log.} \frac{\text{sin.} \frac{1}{2}(x-a)}{\text{cos.} \frac{1}{2}(x+a)}$. Q. E. D.

Theorema III.

$$\int \frac{dx}{\text{cos.}\, x + \text{cos.}\, a} = \frac{1}{\text{sin.}\, a} \text{log.} \frac{\text{cos.} \frac{1}{2}(x-a)}{\text{cos.} \frac{1}{2}(x+a)}.$$

Dem. Quum sit ex trigonometrica angulorum analysi $\text{cos.}\, x + \text{cos.}\, a = 2 \,\text{cos.} \frac{1}{2}(x+a) \,\text{cos.} \frac{1}{2}(x-a)$, erit iccirco $\frac{1}{\text{cos.}\, x + \text{cos.}\, a} = \frac{1}{2} \cdot \frac{1}{\text{cos.} \frac{1}{2}(x+a) \text{cos.} \frac{1}{2}(x-a)}$. Resolvatur

haec

haec fractio in duas $\frac{A\,\text{sin.}\frac{1}{2}(x+a)}{\text{cos.}\frac{1}{2}(x+a)}+\frac{B\,\text{sin.}\frac{1}{2}(x-a)}{\text{cos.}\frac{1}{2}(x-a)}$; hae reducantur ad communem denominatorem, & fiet prioris numerator $\frac{1}{2}=A\,\text{sin.}\,\frac{1}{2}(x+a)\,\text{cos.}\,\frac{1}{2}(x-a)+B\,\text{sin.}\,\frac{1}{2}(x-a)\,\text{cos.}\,\frac{1}{2}(x+a)$. Hic autem animadverto, sumpto $A=-B$ prodire $\frac{1}{2}=A\left[\text{sin.}\frac{1}{2}(x+a)\,\text{cos.}\,\frac{1}{2}(x-a)-\text{sin.}\frac{1}{2}(x-a)\,\text{cos.}\,\frac{1}{2}(x+a)\right]=A\,\text{sin.}\,\frac{1}{2}\left[(x+a)-(x-a)\right]=A\,\text{sin.}\,a$; indeque $A=\frac{1}{2\,\text{sin.}\,a}$, & $B=-\frac{1}{2\,\text{sin.}\,a}$. Quamobrem oritur proposita formula $\frac{dx}{\text{cos.}x+\text{cos.}a}=\frac{1}{\text{sin.}\,a}\cdot\frac{\frac{1}{2}dx\,\text{sin.}\,\frac{1}{2}(x+a)}{\text{cos.}\,\frac{1}{2}(x+a)}-\frac{1}{\text{sin.}\,a}\cdot\frac{\frac{1}{2}dx\,\text{sin.}\,\frac{1}{2}(x-a)}{\text{cos.}\,\frac{1}{2}(x-a)}$, cujus iccirco integratio statim praebet $\int\frac{dx}{\text{cos.}\,x+\text{cos.}\,a}=-\frac{1}{\text{sin.}\,a}\log.\,\text{cos.}\,\frac{1}{2}(x+a)+\frac{1}{\text{sin.}\,a}\log.\,\text{cos.}\,\frac{1}{2}(x-a)=\frac{1}{\text{sin.}a}\log.\frac{\text{cos.}\frac{1}{2}(x-a)}{\text{cos.}\frac{1}{2}(x+a)}$ Q. E. D.

Theorema IV.

$$\int\frac{dx}{\text{cos.}\,x-\text{cos.}\,a}=\frac{1}{\text{sin.}\,a}\log.\frac{\text{sin.}\frac{1}{2}(x+a)}{\text{sin.}\frac{1}{2}(a-x)}.$$

Dem. Theoremata angulorum trigonometrica, praebent $\text{cos.}x-\text{cos.}\,a=2\,\text{sin.}\,\frac{1}{2}(a+x)\,\text{sin.}\,\frac{1}{2}(a-x)$. Igitur $\frac{1}{\text{cos.}x-\text{cos.}a}$ =

$$= \frac{1}{2\,\text{fin.}\frac{1}{2}(a+x)\text{fin.}\frac{1}{2}(a-x)} = \frac{A\,\text{cof.}\frac{1}{2}(a+x)}{\text{fin.}\frac{1}{2}(a+x)} + \frac{B\,\text{cof.}\frac{1}{2}(a-x)}{\text{fin.}\frac{1}{2}(a-x)},$$

& facta reductione ad eundem denominatorem, tumque aequatis utriusque membri numeratoribus prodit $\frac{1}{2} = A\,\text{cof.}\,\frac{1}{2}\,(a+x)\,\text{fin.}\,\frac{1}{2}\,(a-x) + B\,\text{cof.}\,\frac{1}{2}\,(a-x)\,\text{fin.}\,\frac{1}{2}\,(a+x)$. Jamvero palam est, capto $A = B$ fieri $\frac{1}{2} = A\,\text{fin.}\,\frac{1}{2}\left[(x+a)+(a-x)\right] = A\,\text{fin.}\,a$; unde oritur $A = B = \frac{1}{2\,\text{fin.}\,a}$. Quocirca differentialis formula fit $\frac{dx}{\text{cof.}\,x - \text{cof.}\,a} = \frac{1}{\text{fin.}\,a} \cdot \frac{\frac{1}{2}dx\,\text{cof.}\frac{1}{2}(a+x)}{\text{fin.}\,\frac{1}{2}(a+x)} + \frac{1}{\text{fin.}\,a} \cdot \frac{\frac{1}{2}dx\,\text{cof.}\frac{1}{2}(a-x)}{\text{fin.}\,\frac{1}{2}(a-x)}$, sumptisque integralibus invenitur continuo $\int \frac{dx}{\text{cof.}\,x - \text{cof.}\,a} = \frac{1}{\text{fin.}\,a}\log.\text{fin.}\frac{1}{2}(a+x) - \frac{1}{\text{fin.}\,a}\log.\text{fin.}\frac{1}{2}(a-x) = \frac{1}{\text{fin.}\,a}\log.\frac{\text{fin.}\frac{1}{2}(a+x)}{\text{fin.}\frac{1}{2}(a-x)}$.

Q. E. D.

Scholion Generale.

Sumpto angulo quocumque ϕ habetur ex Doctrina functionum angularium $\text{fin.}\,\frac{1}{2}\,\phi = \sqrt{\frac{1-\text{cof.}\phi}{2}}$, & $\text{cof.}\,\frac{1}{2}\,\phi = \sqrt{\frac{1+\text{cof.}\phi}{2}}$. Propterea praecedentes formulae dupliciter exprimuntur:

muntur: I. $\int \frac{dx}{\text{sin.}x + \text{sin.}a} = \frac{1}{\text{cos.}a} \log. \frac{\text{sin.}\frac{1}{2}(x+a)}{\text{cos.}\frac{1}{2}(x-a)} = \frac{1}{2\text{cos.}a} \log. \frac{1-\text{cos.}(x+a)}{1+\text{cos.}(x-a)}$.

II. $\int \frac{dx}{\text{sin.}x - \text{sin.}a} = \frac{1}{\text{cos.}a} \log. \frac{\text{sin.}\frac{1}{2}(x-a)}{\text{cos.}\frac{1}{2}(x+a)} = \frac{1}{2\text{cos.}a} \log. \frac{1-\text{cos.}(x-a)}{1+\text{cos.}(x+a)}$.

III. $\int \frac{dx}{\text{cos.}x + \text{cos.}a} = \frac{1}{\text{sin.}a} \log. \frac{\text{cos.}\frac{1}{2}(x-a)}{\text{cos.}\frac{1}{2}(x+a)} = \frac{1}{2\text{sin.}a} \log. \frac{1+\text{cos.}(x-a)}{1+\text{cos.}(x+a)}$.

IV. $\int \frac{dx}{\text{cos.}x - \text{cos.}a} = \frac{1}{\text{sin.}a} \log. \frac{\text{sin.}\frac{1}{2}(a+x)}{\text{sin.}\frac{1}{2}(a-x)} = \frac{1}{2\text{sin.}a} \log. \frac{1-\text{cos.}(a+x)}{1-\text{cos.}(a+x)}$.

Eulerus *Calc. Int.* Tom. I. §. 261. invenit formulae $\frac{dx}{a+b\,\text{cos.}\,x}$ (si fuerit $a < b$) integrale $= \frac{1}{\sqrt{(bb-aa)}} \log. \frac{a\,\text{cos.}\,x + b + \text{sin.}\,x\sqrt{(bb-aa)}}{a+b\,\text{cos.}\,x}$. Instituta hujus formulae Eulerianae cum nostra Theorematis III. comparatione, nanciscimur $a = \text{cos.}\,a$, $b = 1$, $\sqrt{(bb-aa)} = \text{sin.}\,a$, & consequenter integrale Eulerianum $= \frac{1}{\text{sin.}\,a} \log. \frac{\text{cos.}\,a\,\text{cos.}\,x + \text{sin.}\,a\,\text{sin.}\,x + 1}{\text{cos.}\,x + \text{cos.}\,a} = \frac{1}{\text{sin.}\,a} \log. \frac{1+\text{cos.}(a-x)}{\text{cos.}\,a + \text{cos.}\,x} = \frac{1}{\text{sin.}\,a} \log. \frac{2[\text{cos.}\frac{1}{2}(a-x)]^2}{\text{cos.}\,a + \text{cos.}\,x} = \frac{1}{\text{sin.}\,a} \log. \frac{2[\text{cos.}\frac{1}{2}(a-x)]^2}{2\,\text{cos.}\frac{1}{2}(a-x)\,\text{cos.}\frac{1}{2}(a+x)} = \frac{1}{\text{sin.}\,a} \log. \frac{\text{cos.}\frac{1}{2}(a-x)}{\text{cos.}\frac{1}{2}(a+x)}$, prorsus uti in Theoremate III. demonstratum est.

Si integranda proponatur formula $\frac{d\varphi}{(1+\text{cos.}\,\varphi)^n}$, ubi n numerus est affirmativus integer; adhibita Euleriana substitutione $\text{cos.}\,\varphi = \frac{1-x^2}{1+x^2}$ invenitur integrale prorsus algebraicum. Nam $\text{sin.}\varphi = \frac{2x}{1+x^2}$,

$d\varphi$

$d\phi \operatorname{cos}.\phi = \frac{2dx(1-x^2)}{(1+x^2)^2}$; ideoque $d\phi = \frac{2dx}{1+x^2}$, & denique $\frac{d\phi}{(1+\operatorname{cos}.\phi)^n} = \frac{dx}{2^{n-1}}(1+x^2)^{n-1}$, cujus integrale est manifesto algebraicum, eoque invento sat erit pro x subrogare $\sqrt{\left(\frac{1-\operatorname{cos}.\phi}{1+\operatorname{cos}.\phi}\right)}$, vel tang. $\frac{1}{2}\phi$, vel demum $\frac{\operatorname{sin}.\phi}{1+\operatorname{cos}.\phi}$.

Praeterea cum formulae $dx\sqrt{(1 \pm \operatorname{cos}.x)}$ (facta multiplicatione ac divisione per $\sqrt{(1 \mp \operatorname{cos}.x)}$) integrale se ultro prodat $\pm 2\sqrt{(1 \mp \operatorname{cos}.x)}$; inde haud difficulter infertur, formulae $x^n dx \sqrt{(1 \pm \operatorname{cos}.x)}$ integrale exprimi per hanc seriem $\pm 2x^n$
$\sqrt{(1 \mp \operatorname{cos}.x)}$
$+ 2^2 n x^{n-1} \sqrt{(1 \pm \operatorname{cos}.x)} \mp 2^3 n(n-1) x^{n-2} \sqrt{(1 \mp \operatorname{cos}.x)}$
$- 2^4 n(n-1)(n-2) x^{n-3} \sqrt{(1 \pm \operatorname{cos}.x)} \pm 2^5 n(n-1)(n-2)(n-3) x^{n-4} \sqrt{(1 \mp \operatorname{cos}.x)}$
$+ 2^6 n(n-1)(n-2)(n-3)(n-4) x^{n-5} \sqrt{(1 \pm \operatorname{cos}.x)}$
$\mp 2^7 n(n-1)(n-2)(n-3)(n-4)(n-5) x^{n-6} \sqrt{(1 \mp \operatorname{cos}.x)}$
$- 2^8 n(n-1)(n-2)(n-3)(n-4)(n-5)(n-6) x^{n-7} \sqrt{(1 \pm \operatorname{cos}.x)}$
$+ 2^8 n(n-1)(n-2)(n-3)(n-4)(n-5)(n-6)(n-7) \int x^{n-8} dx \sqrt{(1 \pm \operatorname{cos}.x)}$,
quae manifesto interrumpitur posito x numero integro affirmativo, regiturque lege sat splendida ac simplici. Haud dispar invenitur series pro integrali formulae $x^n dx \sqrt{(1 \pm \operatorname{sin}.x)}$.

Adnotatio ad §. 264.

Cum hoc paragrapho Eulerus invenerit integrale formulae $\frac{d\varphi}{(a+b\,\text{cos}.\,\varphi)^n}$; analogum erit

Problema.

Formulae differentialis $\frac{dx}{(a+b\,\text{tang}.\,x)^n}$; ubi n est numerus integer positivus integrale investigare.

Solutio.

Cum sit $\frac{dx}{(a+b\,\text{tang}.\,x)^n} = \frac{dx}{b^n\left(\frac{a}{b}+\text{tang}.\,x\right)^n}$; facto $\frac{a}{b} = c$ problema reducitur ad integrationem formulae $\frac{dx}{(c+\text{tang}.\,x)^n}$. Sit nunc $c = \text{tang}.\,h$; erit $\frac{dx}{(c+\text{tang}.\,x)^n} = \frac{dx(\text{cos}.\,h.\,\text{cos}.\,x)^n}{(\text{sin}.\,h.\,\text{cos}.\,x+\text{cos}.\,h\,\text{sin}.\,x)^n}$, quod denuo erit (facto $h+x=y$) $= (\text{cos}.\,h)^n\,\frac{dy(\text{cos}.\,y\,\text{cos}.\,h+\text{sin}.\,y\,\text{sin}.\,h)^n}{(\text{sin}.\,y)^n}$. Quare si n erit numerus integer positivus, facta evolutione factoris $(\text{cos}.\,y\,\text{cos}.\,h+\text{sin}.\,y\,\text{sin}.\,h)^n$, problema adducetur ad integrationem terminorum, qui numero finiti erunt, atque habebunt hanc formam B $\frac{dy(\text{cos}.\,y)^m}{(\text{sin}.\,y)^m}$, quorum integrationem docuit Eulerus supra §. 249. reductione secunda.

Bre-

Brevius tamen ipse Fontana reducta formula $\frac{dx}{(a+b\,\text{tang.}x)^n}$ ad formulam $\frac{dx}{(1+h\,\text{tang.}x)^n}$ invenit esse

$$\int\frac{dx}{(1+h\,\text{tang.}x)^n}=\frac{-h}{(n-1)(1+h^2)(1+h\,\text{tang.}x)^{n-1}}$$
$$+\int\frac{2dx}{(1+h^2)(1+h\,\text{tang.}x)^{n-1}}-\int\frac{dx}{(1+h^2)(1+h\,\text{tang.}x)^{n-2}}.$$

Quo facto integrale adducitur ad formulam $\int\frac{dx}{1+h\,\text{tang.}x}$ jam notam.

Adnotatio ad §. 266.

Integratio formularum

$e^{ax}x^n dx$ sin. bx; $e^{ax}x^n dx$ cos. bx

EST $\int e^{ax}x^n dx\,\text{sin.}bx=-\frac{1}{b}e^{ax}x^n\,\text{cos.}bx+\int\frac{a}{b}e^{ax}x^n dx\,\text{cos.}bx$

$$+\int\frac{n}{b}e^{ax}x^{n-1}dx\,\text{cos.}bx$$

$$=-\frac{1}{b}e^{ax}x^n\,\text{cos.}bx+\frac{a}{b^2}e^{ax}x^n\,\text{sin.}bx-\int\frac{a^2}{b^2}e^{ax}x^n dx\,\text{sin.}bx$$
$$-\int\frac{an}{b^2}e^{ax}x^{n-1}dx\,\text{sin.}bx$$
$$+\frac{n}{b^2}e^{ax}x^{n-1}\text{sin.}bx-\int\frac{an}{b^2}e^{ax}x^{n-1}dx\,\text{sin.}bx$$
$$-\int\frac{n(n-1)}{b^2}e^{ax}x^{n-2}dx\,\text{sin.}bx$$

Qua-

Quare erit

$$\int e^{ax} x^n dx \,\text{fin.}\, bx = -\frac{b}{a^2+b^2} e^{ax} x^n \,\text{cof.}\, bx$$
$$+\frac{a}{a^2+b^2} e^{ax} x^n \,\text{fin.}\, bx$$
$$+\frac{n}{a^2+b^2} e^{ax} x^{n-1} \,\text{fin.}\, bx$$
$$-\frac{2an}{a^2+b^2}\int e^{ax} x^{n-1} dx \,\text{fin.}\, bx$$
$$-\frac{n(n-1)}{a^2+b^2}\int e^{ax} x^{n-2} dx \,\text{fin.}\, bx$$

Ex qua aequatione apparet quod si fuerit n numerus positivus integer; hac methodo procedendo devenietur tandem ad formulam summatoriam $A \int e^{ax} dx \,\text{fin.}\, bx =$

$$A\,\frac{e^{ax}(\alpha\,\text{fin.}\, x - \text{cof.}\, x)}{\alpha\alpha+1} + \frac{A}{\alpha\alpha+1}$$ (Eul. hoc §. 266.)

Eadem methodo formula $e^{ax} x^n dx \,\text{cof.}\, bx$ deducitur ad integrationem formulae $B \int e^{ax} dx \,\text{cof.}\, bx =$

$$B\,\frac{e^{ax}(\alpha\,\text{cof.}\, x + \text{fin.}\, x)}{\alpha\alpha+1} + C$$ (§. 270.)

Ad has autem duas formulas

$$e^{ax} x^n dx \,\text{fin.}\, bx ;\; e^{ax} x^n dx \,\text{cof.}\, bx$$

facile revocantur etiam formulae

$$\int e^{ax} x^n dx (\text{fin.} fx)^m \;;\; \int e^{ax} x^n dx (\text{cof.} fx)^m$$

quando m est numerus integer positivus; cum tam $(\text{fin.} fx)^m$, quam $(\text{cof.} fx)^m$ exprimi possit per seriem finitam terminorum, qui sunt formae $B\,\text{fin.}\, bx$, aut $B\,\text{cof.}\, bx$.

Evo-

Evolutio formularum

$$\int \frac{e^{\alpha x} dx}{(\text{fin.}\, x)^n} \quad ; \quad \int \frac{e^{\alpha x} dx}{(\text{cof.}\, x)^n}$$

Licet evolutio, quam hic tradituri sumus non absolvatur nisi per series; tamen quia non statim occurrit, omittendam non duximus.

Ac in primis quoniam est (A) . . . .

$$\int \frac{e^{\alpha x} dx}{(\text{fin.}\, x)^n} = -\frac{e^{\alpha x}(\alpha\, \text{fin.}\, x + (n-2)\, \text{cof.}\, x)}{(n-2)(n-1)} + \frac{\alpha\alpha + (n-2)^2}{(n-2)(n-1)} \int \frac{e^{\alpha x} dx}{(\text{fin.}\, x)^{n-2}}$$

perpetuo deprimetur potentia ipsius fin. x; atque si n fuerit numerus par, deveniemus ad formulam $\int \frac{e^{\alpha x} dx}{(\text{fin.}\, x)^2}$; si vero fuerit impar, remanebit formula $\int \frac{e^{\alpha x} dx}{\text{fin.}\, x}$. Pro his autem duabus formulis nihil juvat aequatio (A), in qua tam si fiat $n = 2$, quam si fiat $n = 1$; resultat valor infinitus in secundo membro aequationis. Hujusmodi tamen valor, qui apparet sub infiniti specie excitat suspicionem an forte per integrationes logarithmicas negotium absolvi debeat. Ac revera est

$$\int \frac{e^{\alpha x} dx}{(\text{fin.}\, x)^2} = -e^{\alpha x} \text{cotang.}\, x + \int \alpha e^{\alpha x} dx\, \text{cotang.}\, x$$

$$= -e^{\alpha x} \text{cotang.}\, x + \alpha e^{\alpha x} \log.\, \text{fin.}\, x - \int \alpha^2 e^{\alpha x} dx \log.\, \text{fin.}\, x$$

Est autem $l.\, \text{fin.}\, x =$

$$-l\, 2 - \text{cof.}\, 2x - \frac{1}{2}\text{cof.}\, 4x - \frac{1}{3}\text{cof.}\, 6x - \frac{1}{4}\text{cof.}\, 8x - \&c.$$

Quare cum sit $\int e^{\alpha x} dx \operatorname{cof.} \lambda x = e^{\alpha x} \cdot \frac{\alpha \operatorname{cof.} \lambda x + \lambda \operatorname{fin.} \lambda x}{\alpha\alpha + \lambda\lambda}$ erit

$$\int \frac{e^{\alpha x} dx}{(\operatorname{fin.} x)^2} = C - e^{\alpha x} \operatorname{cotang.} x + \alpha e^{\alpha x} l. \operatorname{fin.} x$$

$$+ \alpha e^{\alpha x} l 2 + \alpha^2 e^{\alpha x} \quad \frac{\alpha \operatorname{cof.} 2x + 2 \operatorname{fin.} 2x}{\alpha\alpha + 2^2}$$

$$+ \frac{\alpha^2 e^{\alpha x}}{2} \cdot \frac{\alpha \operatorname{cof.} 4x + 4 \operatorname{fin.} 4x}{\alpha\alpha + 4^2}$$

$$+ \frac{\alpha^2 e^{\alpha x}}{3} \cdot \frac{\alpha \operatorname{cof.} 6x + 6 \operatorname{fin.} 6x}{\alpha\alpha + 6^2}$$

$$+ \&c.$$

Est etiam

$$\int \frac{e^{\alpha x} dx}{\operatorname{fin.} x} = e^{\alpha x} l. \operatorname{tang.} \frac{1}{2} x - \int \alpha e^{\alpha x} dx\, l. \operatorname{tang.} \frac{1}{2} x$$

Est autem $l. \operatorname{tang.} \frac{1}{2} x =$

$$- 2 \operatorname{cof.} x - \frac{2}{3} \operatorname{cof.} 3x - \frac{2}{5} \operatorname{cof.} 5x - \frac{2}{7} \operatorname{cof.} 7x - \&c.$$

Erit ergo

$$\int \frac{e^{\alpha x} dx}{\operatorname{fin.} x} = K + e^{\alpha x} l. \operatorname{tang.} \frac{1}{2} x$$

$$+ 2 \alpha e^{\alpha x} \quad \frac{\alpha \operatorname{cof.} x + \operatorname{fin.} x}{\alpha\alpha + 1}$$

$$+ \frac{2 \alpha e^{\alpha x}}{3} \cdot \frac{\alpha \operatorname{cof.} 3x + 3 \operatorname{fin.} 3x}{\alpha\alpha + 3^2}$$

$$+ \frac{2 \alpha e^{\alpha x}}{5} \cdot \frac{\alpha \operatorname{cof} 5x + 5 \operatorname{fin.} 5x}{\alpha\alpha + 5^2}$$

$$+ \&c.$$

Secunda formula $\int \frac{e^{ax} dx}{(\text{cof.} x)^n}$ facile reducitur ad primam facto $x = 90^\circ - y$, unde cof. $x =$ fin. y.

Adnotatio altera ad Sectionem III. Vol. I.

Si perpendiculum demiffum ab initio abfciffarum x in tangentem curvae vocetur P; radius vector $= \sqrt{(x^2 + y^2)}$ vocetur u; cofinus anguli radii vectoris, atque axis abfciffarum, feu cofinus anomaliae $= \frac{x}{\sqrt{(x^2+y^2)}}$ vocetur z; adeo ut ejus finus fit $\sqrt{(1 - z^2)} = \frac{y}{\sqrt{(x^2+y^2)}}$; fit vero V functio radii vectoris u, & Z alia functio ipfius z; quotiefcumque fuerit $P = \frac{u V Z}{\sqrt{(1 + V^2 Z^2)}}$ haberi poterit aequatio curvae per functiones ordinatarum x & y, quemadmodum docuimus in fuperiore Adnot. ad hanc eandem Sectionem ope fubftitutionis Cel. Paoli.

Cum vero in eandem fubftitutionem diligentius inquirerem, utpote quae problemata trajectoriarum non parum juvare poffe videretur; animadverti pro pluribus aliis relationibus propofitis inter perpendiculum P, radium vectorem, & finum, aut cofinum anguli anomaliae haberi trajectorias combinando novam fubftitutionem Paoli cum methodis jam cognitis integrandi aequationes differentiales primi ordinis; atque adeo plures in pofterum haberi poffe, quo methodus integrandi eofdem aequationes primi ordinis ulterius promovebitur.

Revera cum fit perpendiculum $P = \frac{x dy - y dx}{\sqrt{(dx^2 + dy^2)}}$, ac per fubftitutiones $x = uz$; $y = u\sqrt{(1 - z^2)}$ habitum fuerit

rit $\frac{-u^2dz}{\sqrt{(u^2dz^2+(1-z^2)du^2)}}=P$; five $\frac{dz}{du}=\frac{P\sqrt{(1-z^2)}}{u\sqrt{(u^2-P^2)}}$, ac facto $P=uQ$ habeatur $\frac{udz}{du\sqrt{(1-z^2)}}=\frac{Q}{\sqrt{(1-Q^2)}}$; fi fiat $\frac{Q}{\sqrt{(1-Q^2)}}=VZR$, ubi V eft functio folius u, & Z functio folius z; R vero eft functio utriufque u, & z; fiat quoque $\int\frac{dz}{Z\sqrt{(1-z^2)}}=\mu$; $\int\frac{Vdu}{u}=\nu$; habebitur $\frac{d\mu}{d\nu}=R$. Quare fi R erit talis functio ipfarum μ, & ν, ut aequatio $\frac{d\mu}{d\nu}=R$ poffit integrari; habebitur trajectoria.

Primus veluti cafus eft quando $R=1$; tunc habemus $d\mu=d\nu$, five $\frac{dz}{Z\sqrt{(1-z^2)}}=\frac{du\,V}{u}$, quem cafum obfervavimus in priori adnotatione ad hanc Sectionem. Alii cafus habentur ex methodis communibus integrandi aequationem primi ordinis $d\mu=R\,d\nu$.

Ut ergo exhibeamus problema maxime generale, quod folvi poffit per fuperiores fubftitutiones; fit finus anguli, quem curva facit cum radio vectore $=S$; erit $S=\frac{P}{u}=Q=\frac{VZR}{\sqrt{(1+V^2Z^2R^2)}}$, ac proinde tangens hujus anguli $=VZR$. Quotiefcumque ergo in his aequationibus erit R talis functio ipfarum $\mu=\int\frac{dz}{Z\sqrt{(1-z^2)}}$ & $\nu=\int\frac{Vdu}{u}$, ut integrari poffit aequatio $d\mu=R\,d\nu$ habebitur trajectoria.

Notandus eft hoc loco modus habendi trajectoriam cum

P = V;

$P = V$; ubi V eſt functio quaecumque ipſius u. Si enim ſit angulus, quem radius vector $u = \sqrt{(x^2 + y^2)}$ facit cum abſciſſa x vocetur φ; erit $\frac{dz}{\sqrt{(1 - z^2)}} = \frac{P\,du}{u\sqrt{(u^2 - P^2)}} = d\varphi = \frac{V\,du}{u\sqrt{(u^2 - V^2)}}$. Si radius oſculator curvae vocetur r; erit $\frac{u\,du}{r} = dP$, quod per facilem conſtructionem geometricam demonſtratur (Vide Nova Acta Petrop. ad an. 1786. Tom. IV. pag. 106.). Quare ſi r ſit functio quaecumque radii vectoris, ſeu quod idem eſt diſtantiae puncti curvae a puncto quocumque fixo; ſeu $r = U$; habebitur primo aequatio inter perpendiculum P, & radium vectorem, nempe $P = \int \frac{u\,du}{U} = V$, deinde alia inter angulum φ, & u, nempe $\varphi = \int \frac{V\,du}{u\sqrt{(u^2 - V^2)}}$, ut invenit Celeb. Nicolaus Fuss in eleganti Comment. ſupracit. ubi ſi ſit $U = nu^m$, facto $a = \sqrt[2 \cdot m]{n(2-m)}$ reperit

$$u = a \sqrt[1 \cdot m]{\operatorname{cof.}(m-1)\varphi} = a \sqrt[1 \cdot m]{\operatorname{cof.}(1-m)\varphi}.$$

Quare ſi ſit $m = -1$; $n = \frac{1}{3}$; erit $V = u^3$ integrale peculiare, ubi nulla additur conſtans, ſeu curva Cl. Paoli, in qua $P = u^3$, in qua proinde radius curvaturae eſt in ratione inverſa diſtantiae puncti curvae a puncto fixo illuminante ſecundum eas conditiones, quas in problemate Optico aſſumere libuit; ſeu facta hac diſtantia $= D$ eſt $r = \frac{1}{3D}$.

Subſti-

Substitutis valoribus $m = -1$; $n = \frac{1}{3}$; aequatio

$$u = a\sqrt[1-m]{\text{cos.}(1-m)\phi}$$

evadit $u = \sqrt{\text{cos.}\, 2\,\phi}$, seu $u^2 = \text{cos.}\, 2\,\phi$, ad quam simplicissimam curvae quaesitae aequationem devenerat etiam Cl. Paoli Opusc. 4. pag. 173.

Regula allata pag. 71. superiorum Adnotationum intelligenda est intra limites regularum de solutionibus partialibus aequationum, neque pro generaliter vera haberi potest.

F I N I S.

	Errata	*Corrige*
Pag. 6 lin. 8	$\int \frac{dx\,\text{sin.}\,n\,lx}{lx} = \ldots\ldots$	$\int \frac{dx\,\text{sin.}\,n\,lx}{lx} = \text{A tang.}\,n$